Stability of a One Legged Robot using μ-Synthesis

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Nomenclature

L_{c1y}: Distance to center of mass of link 1.
L_{c1x}: Distance to center of mass of link 1 perpendicular to center line.
L_{c2}: Distance to center of mass of link 2.
L_1: Length of link 1.
M_1: Mass of link 1.
M_2: Mass of link 2.
I_1: Moment of Inertia of link 1 about center of mass.
I_2: Moment of Inertia of link 2 about center of mass.
I_m: Moment of Inertia of transmission, pulleys, and belt.
n: Transmission reduction ratio.
$\mathbf{H}(\boldsymbol{\theta})$: 2x2 Inertia matrix.
$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$: 2x2 Coriolis matrix.
$\mathbf{G}(\boldsymbol{\theta})$: 2x1 Gravity vector.
θ_1: Angle of link 1 relative to horizontal. + Counterclockwise (CCW).
θ_2: Angle of link 2 relative to link 1. + CCW.
$\boldsymbol{\theta}$: 2x1 vector of angles.
θ_{1o}: Equilibrium angle for link 1 relative to horizontal.
θ_{2o}: Equilibrium angle for link 2 relative to link 1.
$\boldsymbol{\theta}_o$: 2x1 vector of equilibrium angles.
T: Torque provided by transmission. + CCW.
T_{eq}: Equilibrium torque.
$\boldsymbol{\tau}$: 2x1 Input Torque vector.
t: 2x1 Input Torque vector to linearized equations of motion.
x: 4x1 vector of states of linear system.
u: Input to linear system.
y: Output of linear system.
G_1: Transfer Function from t2 to x1.
G_2: Transfer Function from t2 to x2.
$D\{2\}$: 2x1 vector of Disturbance Torques to the links.
$E\{2\}$: 2x1 vector of Errors.
$\mathcal{K}$: Set of controllers which internal stabilize closed loop system.

Abstract

The focus of this paper is to investigate the feasibility of stabilizing an underactuated double inverted pendulum. This preliminary investigation is motivated by the desire to construct a two-legged walking robot. The pendulum represents a single leg in a walking robot. The design of a stabilizing controller using a μ-synthesis approach is explained in detail and conditions for controllability and observability of the system are given.

1 Introduction

Developing a two legged walking robot has been the topic of research for many years now. A particular interest has been expressed in the construction of a two legged and two armed robot that when coupled to a human can be used to augment his strength [5]. Thus a human wearing this robotic suit can lift heavy objects with ease [6]. Many of the walking robots developed have large feet in order to allow substantial torque to be generated by motors located at the ankles. These large feet limit the distance the robot can approach objects. This problem is known as "ski feet". One approach to avoid this problem is to balance the system with only actuation at the knee and leave the ankle unactuated. Thus a pivoted or cylindrical foot could be used. This paper investigates designing a controller, using the μ-synthesis approach, for one leg of a walking robot.

2 Model

Figure 1 shows the schematic of the model for the under actuated leg. It consists of two rigid links connected by a rotary joint. Link 1 is pivoted at the base so as to allow rotary motion but no translational motion. There is no actuation at this pivot point. A

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DC motor housed in link 1 supplies torque to the rotary joint connecting link 1 and link 2. The torque is transferred from the motor to the transmission by a belt and pulley system. The output torque of the transmission is increased by the transmission ratio n. There are two degrees of freedom (θ_1 and θ_2) of which only one is powered (θ_2). Thus there is one less actuator than degrees of freedom. Such systems are called under actuated [1].

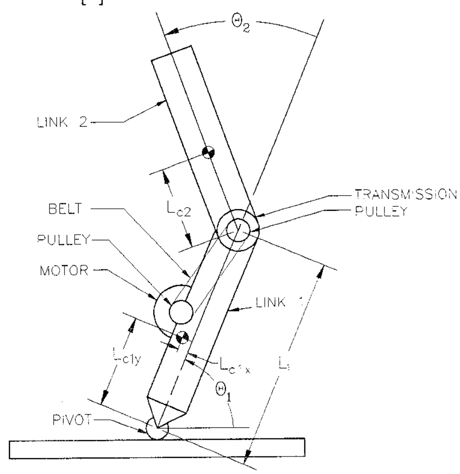

Figure 1: Schematic of the Under Actuated Leg

The dynamic equations of motion, in the absence of any frictional forces are

$$\mathbf{H}(\boldsymbol{\theta})\,\ddot{\boldsymbol{\theta}} + \mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \mathbf{G}(\boldsymbol{\theta}) = \boldsymbol{\tau} \qquad (1)$$

where

$$\begin{aligned}
H_{11}(\theta_2) &= I_1 + I_2 + M_1(L_{c1x}^2 + L_{c1y}^2) \\
&\quad + M_2(L_1^2 + L_{c2}^2 + 2L_1L_{c2}\cos(\theta_2)) \\
H_{12}(\theta_2) &= I_2 + M_2(L_{c2}^2 + L_1L_{c2}\cos(\theta_2)) \\
H_{21}(\theta_2) &= H_{12}(\theta_2) \\
H_{22} &= I_2 + n^2 I_m + M_2 L_{c2}^2 \\
C_{11}(\theta_2,\dot{\theta}_1) &= -2M_2L_1L_{c2}\sin(\theta_2)\dot{\theta}_1 \\
C_{12}(\theta_2,\dot{\theta}_2) &= -M_2L_1L_{c2}\sin(\theta_2)\dot{\theta}_2 \\
C_{21}(\theta_2,\dot{\theta}_1) &= -M_2L_1L_{c2}\sin(\theta_2)\dot{\theta}_1 \\
C_{22} &= 0 \\
G_1(\theta_1,\theta_2) &= M_1g(L_{c1y}\cos(\theta_1) + (L_{c1x}\sin(\theta_1)) \\
&\quad + M_2gL_1\cos(\theta_1) \\
&\quad + M_2gL_{c2}\cos(\theta_1+\theta_2) \\
G_2(\theta_1,\theta_2) &= M_2gL_{c2}\cos(\theta_1+\theta_2) \\
\tau_1 &= 0, \qquad \tau_2 = T
\end{aligned}$$

It is desired to control the system about an operating point where the center of mass of the system is above the pivot point. Equilibrium is defined as the configuration where $\ddot{\theta}_1 = \ddot{\theta}_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$ in equation 1 which leads to

$$\cos(\theta_{1o}+\theta_{2o}) = -\frac{(M_1L_{c1y}+M_2L_1)\cos(\theta_{1o}) + M_1L_{c1x}sin(\theta_{1o})}{M_2L_{c2}} \qquad (2)$$

$$M_2gL_{c2}\cos(\theta_{1o}+\theta_{2o}) = T_{eq} \qquad (3)$$

From equation 2, given a particular θ_{1o}, the corresponding θ_{2o} can be calculated. This equation gives the family of equilibrium points which the leg can be regulated about. Equation 3 gives the equilibrium torque that is needed to be supplied by the actuator.

Since the system is to be controlled in a standing position and not tracking, linear equations of motion about an equilibrium point are needed. Linearizing equation 1 about $(\theta_{1o},\theta_{2o},T_{eq})$ gives

$$\mathbf{h}(\boldsymbol{\theta}_o)\,\ddot{\mathbf{x}} + \mathbf{g}(\boldsymbol{\theta}_o)\,\mathbf{x} = \mathbf{t} \qquad (4)$$

where

$$\begin{aligned}
x_1 &= \theta_1 - \theta_{1o}, & x_2 &= \theta_2 - \theta_{2o} \\
h_{11} &= H_{11}(\theta_{1o}), & h_{12} &= H_{12}(\theta_{2o}) \\
h_{21} &= h_{12}, & h_{22} &= H_{22} \\
g_1 &= M_1g(L_{c1y}\sin(\theta_1) + (L_{c1x}\cos(\theta_1)) \\
&\quad + M_2gL_1\sin(\theta_1) \\
&\quad + M_2gL_{c2}\sin(\theta_1+\theta_2) \\
g_2 &= M_2gL_{c2}\sin(\theta_1+\theta_2) \\
t_1 &= 0, & t_2 &= T - T_{eq}
\end{aligned}$$

This linear equation can be put into familiar state space form by defining

$$x_3 = \dot{\theta}_1, \quad x_4 = \dot{\theta}_2, \quad u = t_2.$$

The state space form is then

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad \mathbf{y} = \mathbf{C}\mathbf{x} \qquad (5)$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{h_{22}g_{11}-h_{12}g_{12}}{\Delta} & \frac{h_{22}g_{12}-h_{12}g_{12}}{\Delta} & 0 & 0 \\ \frac{h_{11}g_{12}-h_{12}g_{11}}{\Delta} & \frac{h_{11}g_{12}-h_{12}g_{12}}{\Delta} & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ -\frac{h_{12}}{\Delta} \\ \frac{h_{11}}{\Delta} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\Delta = h_{11}h_{22} - h_{12}^2.$$

Formulating the problem in this form allows investigation on how the physical parameters affect the system. In designing such a system one must know the conditions under which the system is controllable and observable. It can be shown that the system is observable at all points characterized by equations 2 and 3. On the other hand, it can be determined that the system is always controllable if and only if

$$h_{11}g_{12} - h_{12}g_{11} \neq 0. \tag{6}$$

Thus the system is always controllable as long as the physical parameters are chosen such that the above condition is satisfied.

The linear system given by equation 5 as discussed before has one input (u) and two outputs (x_1 and x_2). Thus there are two transfer functions of interest:

$$G_1 = \frac{x_1}{u} = \frac{(s^2 - \frac{g_{12}}{h_{12}})}{s^4 - (A(3,1) + A(4,1))s^2 + det(A)} \tag{7}$$

$$G_2 = \frac{x_2}{u} = \frac{(s^2 - \frac{g_{11}}{h_{11}})}{s^4 - (A(3,1) + A(4,1))s^2 + det(A)} \tag{8}$$

Note that the dependence of transfer functions on s is not denoted throughout the paper.

Both transfer function have two zeros and four poles. Since the numerators and denominators have non odd powers of s, the poles and zeros will be symmetric about the imaginary axis. This fact makes the transfer functions unstable and non minimum phase which can be affirmed from the physics of the system.

3 Limits of Performance

Using traditional SISO ideas, it can be shown that the poles and zeros in the right half plane limit the performance of the system. The system is represented as shown in figure 2 where G_1 and G_2 are defined in equations 7 and 8 and K represents the controller.

The performance is defined in terms of the transfer function from disturbances at the plant output to regulation error. Disturbances d_1 and d_2 to θ_1 and θ_2 are modeled independently. Regulation error is defined to be the deviation of $\boldsymbol{\theta}$ from $\boldsymbol{\theta}_o$. To apply traditional SISO ideas, one can consider the system indicated in the large dashed box as the equivalent controller K_{eq} with stable internal dynamics applied to G_1. Then the transfer function from d_1 to e_1 is

$$S = \frac{e_1}{d_1} = \frac{1}{1 + G_1 K_{eq}} \tag{9}$$

Let z_1 be the right half plane zero, and p_1 and p_2 be the two right half plane poles for G_1. Freudenberg

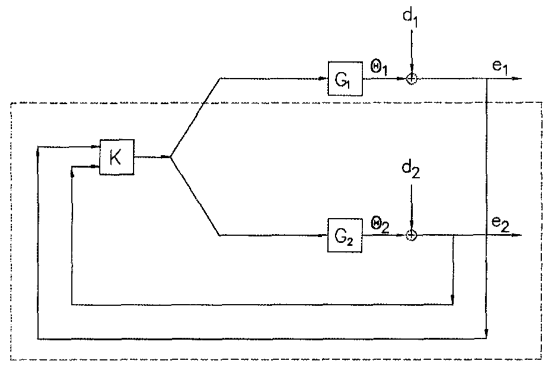

Figure 2: Block Diagram of Closed-loop Linearized System

and Looze [4] have shown that right half poles and zeros impose the following limitations:

$$\int_0^{\infty} \log|S(j\omega)| \frac{2z_1}{z_1^2 + \omega^2} d\omega = \pi log \left| \frac{(p_1 + z_1)(p_2 + z_1)}{(p_1 - z_1)(p_2 - z_1)} \right|. \tag{10}$$

Thus, there is a limit to the performance one can demand from a system with right half plane zero and poles. Equation 10 states that as the zero and the poles get closer together the performance degrades. Also by looking at the complimentary sensitivity function, one can show that the system cannot tolerate much parameter uncertainty in the model near the frequency of the right half plane poles. Since the physical parameters of a walking machine change (ie when a load is picked up), an adaptive control law should be implemented to adjust to the change in parameters. For a detailed analysis of these limitations consult [3, 4]. A similar analysis can be done with G_2 to show limitations imposed on it by the right half plane poles and zeros.

4 Controller Design

The μ-synthesis approach was used to design the controller for the under actuated leg. This approach allows performance objectives to be achieved in the presence of modeling uncertainty which is an important consideration in this system. For further information on this technique consult [2].

Figure 3 shows the interconnection structure used in the design procedure. The solid blocks represent the nominal linear model of the system in equation 4. The weighting functions W_{p1}, W_{p2}, W_{p3} and the norm bounded stable transfer functions Δ_1 and Δ_2 are used to model plant uncertainty. The stable transfer functions Δ_1 and Δ_2 are assumed to be unknown except

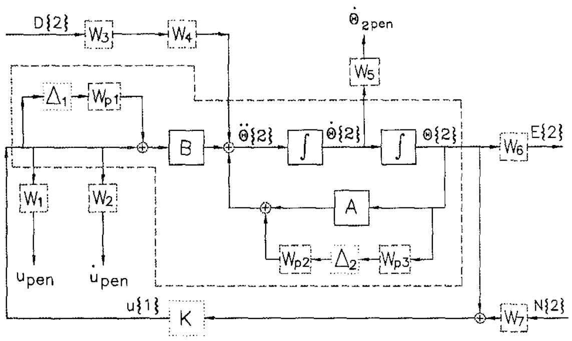

Figure 3: Leg Interconnection Structure

for the norm condition $\|\Delta_{1,2}\|_\infty < 1$. This uncertainty set is denoted $\boldsymbol{\Delta}$. The plant uncertain model set is represented by the large dashed box of Figure 3, where $\Delta_{1,2} \in \boldsymbol{\Delta}$. The frequency dependent weighting functions W_1 through W_7 are used to define performance objectives which are described in detail below. These fictitious weighting functions are used in the controller design only and are not implemented in the actual closed loop system. Only the controller K is actually implemented on the physical system. K is chosen from the set of 2-input, 1-output, real rational matrix functions which internally stabilize the closed-loop system. This set will be denoted $\mathcal{K}$. For any $\Delta_1, \Delta_2 \in \boldsymbol{\Delta}$ choose $K \in \mathcal{K}$, then the closed-loop interconnection in Figure 3 represents a 4-input, 5-output transfer matrix $T(\Delta_1, \Delta_2, K)$ where the four inputs are D{2} and N{2}, and the five outputs are E{2}, u_{pen}, $\dot{u}_{pen}$, and $\dot{\Theta}_{pen}$. These are the input/output signals used to define the performance objective. The goal of the controller design technique is to perform the following optimization

$$\max_{\substack{\Delta_1,\Delta_2 \in \text{stable} \\ \|\Delta_{1,2}\| \leq 1}} \min_{K \in \mathcal{K}} \|T(\Delta_1, \Delta_2, K)\|_\infty$$

This equation shows the importance of the weighting functions in defining a meaningful control design problem. The choice of weighting functions will be discussed below. Details are given in section 6.

The block designated as W_{p1} in Figure 3 is used to model the uncertainty in the actual torque applied to the system as compared to output of the controller. In this system, there is a belt and pulley system which is used to transfer power from the motor to the transmission. The belt and pulley system add extra dynamics to the output of the actuator. This weighting function allows for this extra dynamics to affect the controller design process.

The blocks designated as W_{p2} and W_{p3} are used to model the uncertainty in the physical parameters. A multiplicative model for uncertainty was used. The uncertainities in the individual parameters are all lumped together. Thus the uncertainty is only in the individual entries of the matrix A.

The block designated as W_1 is used to penalize the output from the controller. Since this is a physical system, there is a limit to the amount of torque that can be produced from the motor. This block incorporates the saturation and limits the bandwidth of the actuator in the design process. Since the time domain signal from the controller in the design process is of order 1, the weighting function should be chosen so that $\|W_1\|_\infty \leq \frac{1}{saturation}$.

The block designated as W_2 is used to penalize the derivative of the output signal from the controller. This penalty allows the designer to control the rate at which the controller output can change.

The block designated as W_3 is used to characterize the disturbance torques the system will experience during operation. The D{2} time domain signals here are of order 1. Hence, the large magnitude weighting functions correspond to large magnitude disturbances. There are two weighting functions here since there are assumed to be two disturbances. One disturbance acting on link 1, and one acting on link 2.

The block designated as W_4 is not used to penalize or weight anything. It is used to orient the disturbance torques so they enter the system properly. Since a disturbance on link 1 will not affect link 2 and a disturbance on link 2 will impose a equal but opposite disturbance on link 1, W_4 was chosen as below.

$$W_4 = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{11}$$

The block designated as W_5 is used to penalize the angular velocity of the second link. Penalizing $\dot{\theta}_2$ indirectly decreases the maximum excursion θ_2 undergoes. The larger the penalty, the slower link 2 will move when it leaves the equilibrium point, and the smaller the excursion from equilibrium θ_2 will undergo.

The block designated as W_6 is used to set the performance objective. By adjusting the magnitude of this weighting function, the tracking error in angles can be specified. For example, if the weighting function for θ_1 was set to 100 at DC, then the controller would be designed to give 1% tracking at DC. Thus this weight function is used to set the performance objectives.

The block designated as W_7 is used to model the corruption of each measurement with sensor noise. The noise was assumed to be additive.

This interconnection structure was used to iteratively design a controller using the μ-Analysis and Synthesis Toolbox for Matlab.

5 Experimental Results

A physical system of the leg was built at University of California at Berkeley. A picture is shown below in Figure 4. The physical parameters for the ac-

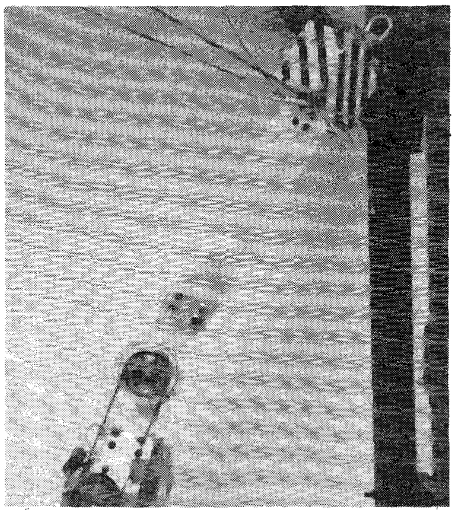

Figure 4: Physical System built at UC Berkeley

tual leg, based on the model, are as follows: $L_{c1y} = 0.298$m, $L_{c1x} = 0.008$m, $L_{c2} = 0.304$m, $L_1 = 0.508$m, $M_1 = 17.007$kg, $M_2 = 8.174$kg, $I_1 = 0.559$kgm^2, $I_2 = 0.390$kgm^2, $I_m = 0.0020$kgm^2, and n = 60.

Since the H_∞ optimal control design was used, a set of weighting functions as described above had to be chosen. The weighting functions are

$$W_1 = \frac{4s+14.6}{s+194}, \quad W_2 = \frac{5s}{s+5}$$
$$W_{p1} = \frac{5s+74}{s+1472}, \quad W_5 = \frac{5s+63.2}{s+253}$$
$$W_3 = \begin{bmatrix} \frac{0.001s+.5}{50s+1} & 0 \\ 0 & \frac{.001s+.5}{10s+1} \end{bmatrix}$$
$$W_{p2} = \begin{bmatrix} 4\% & 4\% & 0 & 0 \\ 0 & 0 & 4\% & 4\% \end{bmatrix}$$
$$W_{p3}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$
$$W_6 = \begin{bmatrix} \frac{0.5s+8.66}{s+0.11} & 0 \\ 0 & \frac{.5s+0.867}{s+0.043} \end{bmatrix}$$
$$W_7 = \begin{bmatrix} 0.000125 & 0 \\ 0 & 0.000025 \end{bmatrix}$$

W_{p1} was chosen such that at low frequencies there is only 5% uncertainty in the torque delivered to the transmission and 100% at high frequencies where the crossover frequency was set at $300\frac{rad}{sec}$. The crossover was set here since $375\frac{rad}{sec}$ is the frequency which excites the first mode of the belt. Thus higher order dynamics of the belt enter close to this frequency and should be noted.

W_{p2} and W_{p3} were chosen such that there is a 4% multiplicative uncertainty in A(3,1), A(3,2), A(4,1), and A(4,2).

W_1 was chosen such that at low frequencies the controller output is limited to 13.3Nm, and at high frequencies the output is limited to .25Nm. These bounds were chosen such that at low frequencies the command input to the servo amplifier would remain in its linear range. At high frequencies which would typically occur if the system went unstable, the gain was chosen so no damage would occur to the system. The crossover frequency was set to $50\frac{rad}{sec}$ well above normal operation.

W_2 was chosen such that at low frequencies it would be a differentiator with a gain of 5 and level off at $5\frac{rad}{sec}$.

W_3 was chosen such that at low frequencies the system could reject a disturbance torque of 0.5Nm on both links. The cutoff frequencies for both were set to $0.01\frac{rad}{sec}$.

W_4 was chosen such that at low frequencies $\dot{\theta}_2$ is limited to 4 $\frac{rad}{sec}$, and at high frequencies is limited to $0.2\frac{rad}{sec}$. The cross over frequency was set to $50\frac{rad}{sec}$.

W_6 was chosen such that at low frequencies there is only 1.25% error in tracking in θ_1 and 5% error in tracking in θ_2. The crossover frequencies were set to $10\frac{rad}{sec}$ and $1\frac{rad}{sec}$ for θ_1 and θ_2 respectively. At higher frequencies the performance specifications were relaxed to 200% error.

W_7 was chosen to be the smallest unit of measurement for both devices. Since both devices are high precision encoders, this bound is reasonable.

With these weighting function, an 18th order controller was developed which achieved a μ of 0.966. Thus, robust performance and robust stability are guaranteed on the linear plant.

Figure 5 shows the real time response of the system subject to the controller designed above. Both a simulated (dashed) and real(solid) response are shown. In the actual system, the initial deviations from equilibrium are smaller. However, the secondary deviations are larger. This effect occurs since there is friction in the system which is slowing down the system. Another interesting feature is that the time responses for θ_1 and θ_2 seem symmetrical. This symmetry can be seen from equation 4. The system must satisfy this equation to be in equilibrium. Thus this equation constrains the angles such that if θ_1 moves counter-

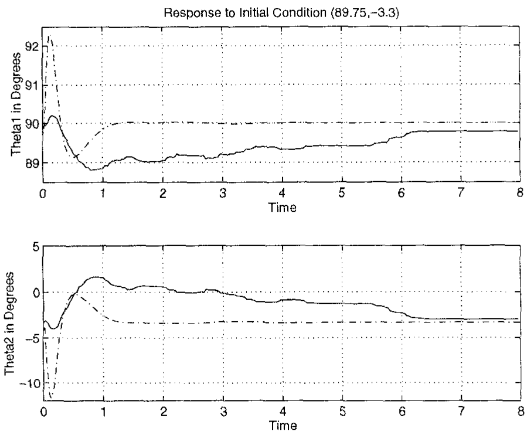

Figure 5: Simulated (dashed) and Actual(solid) System Response

clockwise, θ_2 must move clockwise to keep the center of mass over the pivot point. As shown, steady state error exists in both angles. θ_1 settles to 89.78°, and θ_2 settles to −3.15°. Both of these deviations are within the performance specifications set by the W_6 weighting function.

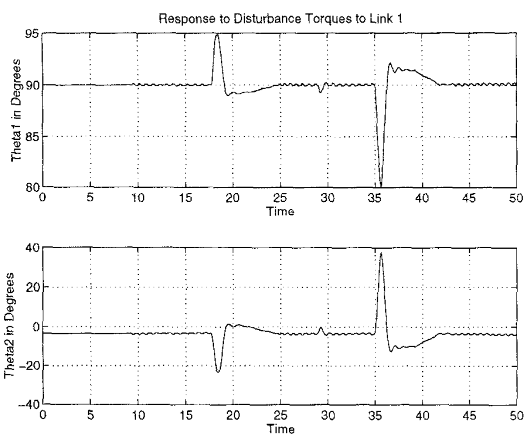

Figure 6: Effect of Disturbance Torque on Link1

The next experiment was to test the system's robustness to disturbances. Figure 6 shows the response of the system to three disturbance torques applied to link 1. In this system, disturbance rejection is defined in a slightly different way. In conventional practice, controllers are designed such that disturbances have no effect on the output. However, since this system is under actuated, the center of mass must move to counteract any disturbance torques imposed on it. Thus the angles will change from the desired value in order to maintain stability. The first disturbance is a torque in the counterclockwise direction with a magnitude of approximately 0.2Nm. It is applied at approximately 17 seconds and then released at 18 seconds. The system then returns to equilibrium in 7 seconds. As noted before, θ_2 moves in the opposite direction of θ_1 such that the center of mass of the system remains above the pivot point. A small disturbance torque is applied at 29 seconds. A torque of approximately 0.5Nm was applied in the clockwise direction at 35 seconds which was the maximum disturbance torque included in the design. The system recovered in approximately 7 seconds again. The same behavior was observed when disturbances were imposed on link 2.

6 Conclusion

A robustly stabilizing controller was designed for a single leg using a μ-synthesis approach. Performance was achieved according to design specifications.

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