Control and Stability Analysis of Cooperating Robots

H. Kazerooni
T. I. Tsay

Mechanical Engineering Department
University of Minnesota
Minneapolis, Minnesota, 55455

Abstract
The work presented here is the description of the control strategy of two cooperating robots. A two-finger hand is an example of such a system. The control method allows for position control of the contact point by one of the robots while the other robot controls the contact force. The stability analysis of two robot manipulators has been investigated using unstructured models for the dynamic behavior of robot manipulators. For the stability of two robots, there must be some initial compliancy in either robot. The initial compliancy in the robots can be obtained by a non-zero sensitivity function for the tracking controller or a passive compliant element such as an RCC.

Introduction
This paper develops the essential rules in stability analysis of two cooperating robots. We assume the robots initially have some type of independent tracking capabilities. This assumption permits us to extend the control analysis to cover industrial robot manipulators in addition to research robots. The tracking capability allows each robot to follow its individual command independently when it is not constrained by each other. Once the robots come in contact with each other, the contact force between the two robots is fed back to one of the robots to develop compliancy (1,2,3,4). The compliancy in one of the robots allows for control of the contact force, while the other robot governs the position of the contact point. A stability bound has been developed on the size of the force feedback gain to stabilize the closed loop system of both robots. The tracking capability allows each robot to follow its individual command independently when it is not constrained by each other. Once the robots come in contact with each other, the contact force between the two robots is fed back to one of the robots to develop compliancy (1,2,3,4). The compliancy in one of the robots allows for control of the contact force, while the other robot governs the position of the contact point. A stability bound has been developed on the size of the force feedback gain to stabilize the closed loop system of both robots. The tracking capability allows each robot to follow its individual command independently when it is not constrained by each other. Once the robots come in contact with each other, the contact force between the two robots is fed back to one of the robots to develop compliancy (1,2,3,4). The compliancy in one of the robots allows for control of the contact force, while the other robot governs the position of the contact point. A stability bound has been developed on the size of the force feedback gain to stabilize the closed loop system of both robots.

Dynamic Model of the Robot
In this section, a general approach will be developed to describe the dynamic behavior of a large class of industrial and research robot manipulators having positioning (tracking) controllers. The fact that most industrial manipulators already have some kind of positioning controller is the motivation behind our approach. Also, a number of methodologies exist for the development of robust positioning controllers for direct and non-direct robot manipulators (5).

In general, the end-point position of a robot manipulator that has a positioning controller is a dynamic function of its input trajectory vector, \( \mathbf{e} \), and the external force, \( \mathbf{f} \). Let \( \mathbf{G} \) and \( \mathbf{S} \) be two \( \mathbf{L}_2 \) stable mappings that describe the dynamic behavior of the robot manipulators. This unified approach of modeling robot dynamics is expressed in terms of sensitivity functions as opposed to the Lagrangian approach. It allows us to incorporate the dynamic behavior of all the elements of a robot manipulator (i.e. actuators, sensors and the structural compliance of the links) in addition to the rigid body dynamics (4).

Dynamics of Two Robots
Suppose two manipulators with dynamic equation 1 are in contact with each other. Equations 3 and 4 represent the entire dynamic behavior of two interacting robots.

\[ y_1 = G_1(e_1) + S_1(f_1) \quad (3) \]
\[ y_2 = S_2^{-1}(y_2 - G_2(e_2)) \quad (4) \]

where: \( y_1, y_2 \) and \( f_1, f_2 \) are calculated from equations 5.

Figure 1 shows the block diagram of the interaction of two robots. Note that the blocks in Figure 1 are in general non-linear operators, however, in the linear case one can treat these blocks as transfer function matrices.

Equation 5 motivates the block diagram of Figure 2 for representation of the contact force in the system where \( V_1 \) and \( V_2 \) are given by equations 6 and 7.

\[ f_2 = \left( S_2^{-1} \right)^{-1} (G_1(e_1) - G_2(e_2)) \quad (5) \]
We assume Figure 2 is valid for representation of the nonlinear case also. In other words, considering equations 3 and 4 as original equations for dynamic behavior of the robots, one can arrive at operators $V_1$ and $V_2$ to show the contributions of $e_1$ and $e_2$ on the contact force. We assume $V_1$ and $V_2$ are two $L_p$-stable operators, in other words $V_1(e_1): L_p^n \rightarrow L_p^n$ and $V_2(e_2): L_p^n \rightarrow L_p^n$ and also there exist positive scalars $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ such that:

$$\|V_1(e_1)\|_p \leq \alpha_1\|e_1\|_p + \beta_1$$  \hspace{1cm} (9)

$$\|V_2(e_2)\|_p \leq \alpha_2\|e_2\|_p + \beta_2$$  \hspace{1cm} (10)

See Appendix A for some definitions on the $L_p$ stability.

The Closed-loop System for Two Robots

The control architecture in Figure 3 shows how we develop compliancy in the system. $H_2$ is a compensator to be designed for the second robot. The input to this compensator is the contact force, $f_2$. The compensator output signal is being added vectorially with the input command vector, $r_2$, resulting in the error signal, $e_2$, for the second robot manipulator. One can think of this architecture as a system that allows the second robot to "control" the force and the first robot to "control" the position.

There are two feedback loops in the system; the first loop (which is the natural feedback loop), is the same as the one shown in Figure 1. This loop shows how the contact force affects the robots in a natural way when two robots are in contact with each other. The second feedback loop is the controlled feedback loop.

If two robots are not in contact, then the dynamic behavior of each robot reduces to the one shown by equation 1 (with $f=0$), which is a simple tracking system. When the robots are in contact with each other, then the contact forces and the end-point positions of the robots are given by $f_1$, $f_2$, $u_1$ and $u_2$ where the following equations are true:

$$u_1 = G_1(e_1) + S_1(f_1)$$  \hspace{1cm} (11)

$$f_2 = S_2^{-1}(u_2 - G_2(e_2))$$  \hspace{1cm} (12)

$$u_1 = u_2$$  \hspace{1cm} (13)

$$f_1 + f_2 = 0$$  \hspace{1cm} (14)

$$e_2 = r_2 + H_2(f_2)$$  \hspace{1cm} (15)

If all the operators are considered linear transfer function matrices, then:

$$f_2 = S_1 + S_2 + G_2 H_2^{-1}(G_1 e_1 - G_2 r_2)$$  \hspace{1cm} (16)

We plan to choose a class of compensators, $H_2$, to control the contact force with the input command $r_2$. This controller must also guarantee the stability of the closed-loop system shown in Figure 3. Note that the robot sensitivity functions and the electronic compliancy, $G_2 H_2$, add together to form the total sensitivity of the system. If $H_2 G_2$, then only the sensitivity functions of two robots add together to form the compliancy for the system. By closing the loop via $H_2$, one can not only add to the total sensitivity but also shape the sensitivity of the system.

When two robots are not in contact with each other, the actual end-point position of each robot is almost equal to its given in Appendix A. The following corollary develops a stability bound if $H_2$ is selected as a linear operator (the impulse response) while all the other operators are still nonlinear, then:

$$\|H_2(f_2)\|_p \leq \gamma \|f_2\|_p$$  \hspace{1cm} (22)

where: $\gamma = \sigma_{max}(N)$  \hspace{1cm} (23)

Stability

The objective of this section is to arrive at a sufficient condition for stability of the system shown in Figures 3. This sufficient condition leads to the introduction of a class of compensators, $H_2$, that can be used to develop compliancy for the class of robot manipulators that have positioning controllers. The following theorem (Small Gain Theorem) (7,8) states the stability condition of the closed-loop system shown in Figure 4. A corollary is given to represent the size of $H_2$ to guarantee the stability of the system.

If conditions I, II and III hold:

I. $V_1$ and $V_2$ are $L_p$-stable operators, that is $V_1(e_1): L_p^n \rightarrow L_p^n$ and $V_2(e_2): L_p^n \rightarrow L_p^n$ and:

a) $\|V_1(e_1)\|_p \leq \alpha_1\|e_1\|_p + \beta_1$  \hspace{1cm} (17)

b) $\|V_2(e_2)\|_p \leq \alpha_2\|e_2\|_p + \beta_2$  \hspace{1cm} (18)

II. $H_2$ is chosen such that mapping $H_2(f_2)$ is $L_p$-stable, that is

a) $H_2(f_2): L_p^n \rightarrow L_p^n$  \hspace{1cm} (19)

b) $\|H_2(f_2)\|_p \leq \alpha_3\|f_2\|_p + \beta_3$  \hspace{1cm} (20)

III. and $\alpha_2\alpha_3 < 1$  \hspace{1cm} (21)

then the closed loop system in Figure 4 is stable. The proof is given in Appendix A. The following corollary develops a stability bound if $H_2$ is selected as a linear transfer function matrix.

Corollary

The key parameter in the proposition is the size of $\alpha_2\alpha_3$. According to the proposition, to guarantee the stability of the system, $H_2$ must be chosen such that $\alpha_2\alpha_3 < 1$. If $H_2$ is chosen as a linear operator (the impulse response) while all the other operators are still nonlinear, then:

$$\|H_2(f_2)\|_p \leq \gamma \|f_2\|_p$$  \hspace{1cm} (22)

where: $\gamma = \sigma_{max}(N)$  \hspace{1cm} (23)
The third stability condition, inequality 21, can be rewritten as:
\[ \gamma \sigma_2 < 1 \] (24)
To guarantee the closed loop stability, \( \gamma \sigma_2 \) must be smaller than unity, or, equivalently:
\[ \frac{1}{\gamma \sigma_2} < 1 \] (25)
To guarantee the stability of the closed loop system, \( H_2 \) must be chosen such its "size" is smaller than the reciprocal of the "size" of the forward loop mapping in Figure 4. Note that \( \gamma \) represents a "size" of \( H_2 \) in the singular value sense.

When all the operators are linear transfer function matrices one can use Multivariable Nyquist Criterion to arrive at the sufficient condition for stability of the closed loop system. This sufficient condition leads to the introduction of a class of transfer function matrices, \( H_2 \), that stabilize the family of linearly treated robot manipulators. The detailed derivation for the stability condition is given in Appendix B. Appendix C shows that the stability condition given by Nyquist Criterion is a subset of the criteria given by the Small Gain Theorem. Using the results in Appendix B, the sufficient condition for stability is given by inequality 26:
\[ \sigma_{max}(H_2) < \frac{1}{\sigma_{max}(\{S_1 + S_2\}^{-1} G_2)} \quad \forall \omega \in (0,\infty) \] (26)
Similar to the nonlinear case, \( H_2 \) must be chosen such that its "size" is smaller than the reciprocal of the "size" of the forward loop mapping in Figure 5 to guarantee the stability of the closed loop system. Note that in inequality 26 \( \sigma_{max} \) represents a "size" of \( H_2 \) in the singular value sense.

Consider \( n=1 \) (one degree of freedom system) for more understanding about the stability criterion. The stability criterion when \( n=1 \) is given by inequality 27.
\[ |G_2 H_2| < |S_1 + S_2| \quad \forall \omega \in (0,\infty) \] (27)
where \(|\cdot|\) denotes the magnitude of a transfer function. Since in many cases \( S_2=1 \) within the bandwidth of the tracking controller of each robot, \( \omega_0 \), then \( H_2 \) must be chosen such that:
\[ |H_2| < |S_1 + S_2| \quad \forall \omega \in (0,\omega_0) \] (28)
Inequality 28 reveals some facts about the size of \( H_2 \). The smaller the sensitivity functions of the robot manipulators are, the smaller \( H_2 \) must be chosen. In the "ideal case", no \( H_2 \) can be found to allow two perfect tracking robots \( S_1, S_2 \) to interact with each others. In other words, for the stability of the system shown in Figure 3, there must be some compliancy in either first or second robot. RCC, structural dynamics, and the tracking controller stiffness form the compliancy on the robot.

The maximum singular value of a matrix \( A \), \( \sigma_{max}(A) \), is defined as:
\[ \sigma_{max}(A) = \max \left\{ \frac{|A z|}{|z|} \right\} \]
where \( z \) is a non-zero vector and ||\cdot|| denotes the Euclidean norm.

Suppose, the first robot is an ideal positioning system. In other words, \( S_1 \) has a zero gain. Therefore the contact force and the position of contact point between two robots are:
\[ f_{2 \infty} = (S_2 + G_2 H_2)^{-1} (G_1 e_1 - G_2 f_2) \]
\[ u_{1 \infty} = G_1 e_1 \]
The first robot controls the position of the contact point, while the other controls the contact force. Generalizing this concept to \( n \) robots, one robot controls the position of the contact point while the other robots control the contact forces such that:
\[ f_1 + f_2 + f_3 + \ldots + f_n = 0 \] (31)

**Summary and Conclusion**
A new architecture for compliance control of two cooperating robots has been investigated using unstructured models for dynamic behavior of robots. Each robot end-point follows its position input command vector "closely" when the robots are not in contact with each other. When two robots come in contact with each other, one robot controls the position of the contact point, while the other controls the contact force. The unified approach of modeling robots is expressed in terms of sensitivity functions. A bound for the global stability of the manipulators has been derived. For the stability of two robots, there must be some initial compliancy in either robot. The initial compliancy in the robots can be obtained by a non-zero sensitivity function for the tracking controller or a passive compliant element such as an RCC.

**Example**
Consider two one-degree of freedom robots with \( G \) and \( S \) in equation 1 given as:
\[ G_1(s) = \frac{1}{(s/5 + 1)(s/9 + 1)(s/190 + 1)(s/240 + 1)(s/290 + 1)} \]
\[ G_2(s) = \frac{1}{(s/6 + 1)(s/10 + 1)(s/200 + 1)(s/250 + 1)(s/300 + 1)} \]
\[ S_1(s) = \frac{0.1}{(s/4 + 1)(s/8 + 1)} \]
\[ S_2(s) = \frac{0.05}{(s/5 + 1)(s/9 + 1)} \]
Both robots have good positioning capability (small gain for \( S \)). The poles that are located at -250, -300, -290, -240 show the high frequency modes in the robots. The stability of the robots when they are in contact with each other is analyzed. If we consider \( H_2 \) as a constant gain, then inequality 27 yields that for \( H_2(0.08) \) the value of \( |G_2 H_2| \) is always smaller than \( |S_1 + S_2| \) for all \( \omega \in (0,\infty) \). Figure 6 shows the plots of \( |G_2 H_2| \) and \( |S_1 + S_2| \) for three values of \( H_2 \). For \( H_2=0.05 \) the system is stable with the closed loop poles located at (-456.71, -147.24; 172.37i), (-9.41, -8.38, -5.62, -4.58) while \( H_2=1 \) results in unstable system with the closed loop poles located at (-400.88, -9.03, -8.05, -5.05, -4.13, 23.98 + 474.35j). Note that the stability condition derived via inequality 27 is a sufficient condition for stability; many compensators can be found to stabilize the system without satisfying inequality 27. Figure 6 shows an example (\( H_2=0.25 \)) that does not satisfy inequality 27 however the system is stable with closed loop poles at (-598.64, -76.87 + 298.04i), (-9.1, -8.15, -5.19, -4.36). If one uses root locus for stability analysis, for \( H_2=0.75 \) all the closed loop poles will be in the left half plane. Once a constant value for stabilizing \( H_2 \) established, one can choose a dynamic compensator to filter out the high frequency noise in the force measurements.

\[ H_2 = 0.05 \]
The resulting output belongs to \( L^p \). Moreover, the norm of the output is not larger than \( \alpha_2 \) times the norm of the input plus the offset constant \( \beta_2 \).

Definition 6: The smallest \( \alpha_2 \) such that there exists a \( \beta_2 \) so that inequality b of Definition 5 is satisfied is called the gain of the operator \( V_2(\cdot) \).

Definition 7: Let \( V_2(\cdot): L^p_{\text{pe}} \rightarrow L^p_{\text{pe}} \). The operator \( V_2(\cdot) \) is said to be causal if:

\[
V_2(e_2) = V_2(e_2_T) \quad \forall T < \infty \quad \text{and} \quad \forall e_2 \in L^p_{\text{pe}}
\]

Proof of the nonlinear stability proposition

Define the closed-loop mapping \( A(e_1,e_2) \) by:

\[
e_2 = e_2 + e_2 \quad (11)
\]

For each finite \( T \), inequality b is true.

\[
\|e_2_T\|_p \leq \|e_1_T\|_p + \|e_2_T\|_p \quad \forall T < \infty
\]

Since \( \alpha_3 \alpha_2 \) is less than unity:

\[
\|e_2_T\|_p < \infty \quad \forall T < \infty
\]

Inequality b shows the linear boundedness of \( e_2 \).

(Condition b of definition 5) Inequality b and a taken together, guarantee that the closed-loop mapping \( A \) is \( L^p \)-stable.

Appendix B

The objective is to find a sufficient condition for stability of the closed-loop system in Figure 5 by Nyquist Criterion. The block diagram in Figure 5 can be reduced to the block diagram in Figure B1 when all the operators are linear transfer function matrices.

\[
\begin{align*}
\text{Appendix A} \\
\text{Definitions 1 to 7 will be used in the stability proof of the closed-loop system (7,8).}
\end{align*}
\]

Definition 1: For all \( p \in (1, \infty) \), we label as \( L^p_{\text{pe}} \) the set consisting of all functions \( f = (f_1,f_2,\ldots,f_n)^T: (0,\infty) \rightarrow \mathbb{R}^n \) such that:

\[
\int_0^\infty f_i(t) dt < \infty \quad \text{for} \quad i = 1,2,\ldots,n
\]

Definition 2: For all \( T \in (0, \infty) \), the function \( f_T \) defined by:

\[
f_T(t) = f(t) \quad 0 \leq t < T
\]

is called the truncation of \( f \) to the interval \( (0, T) \).

Definition 3: The set of all functions \( f = (f_1,f_2,\ldots,f_n)^T: (0,\infty) \rightarrow \mathbb{R}^n \) such that \( f \in L^p_{\text{pe}} \) for all finite \( T \) is denoted by \( L^p_{\text{pe}} \).

Definition 4: The norm on \( L^p_{\text{pe}} \) is defined by:

\[
\|f\|_p = \left( \sum_{i=1}^n \|f_i\|_p^2 \right)^{1/2}
\]

where \( \|f_i\|_p \) is defined as:

\[
\|f_i\|_p = \left( \int_0^\infty w_i(t) f_i(t)^p dt \right)^{1/p}
\]

where \( w_i \) is the weighting factor. \( w_i \) is particularly useful for scaling forces and torques of different units.

Definition 5: Let \( V_2(\cdot): L^p_{\text{pe}} \rightarrow L^p_{\text{pe}} \). We say that the operator \( V_2(\cdot) \) is \( L^p \)-stable, if:

a) \( V_2(\cdot): L^p_{\text{pe}} \rightarrow L^p_{\text{pe}} \)

b) there exist finite real constants \( \alpha_2 \) and \( \beta_2 \) such that:

\[
\|V_2(e_2)\|_p \leq \alpha_2 \|e_2\|_p + \beta_2 \quad \forall e_2 \in L^p_{\text{pe}}
\]

According to this definition we first assume that the operator maps \( L^p_{\text{pe}} \) to \( L^p_{\text{pe}} \). It is clear that if one does not show that \( V_2(\cdot): L^p_{\text{pe}} \rightarrow L^p_{\text{pe}} \), the satisfaction of condition (a) is impossible since \( L^p_{\text{pe}} \) contains \( L^p \). Once the mapping of \( V_2(\cdot) \) from \( L^p_{\text{pe}} \) to \( L^p_{\text{pe}} \) is established, then we say that the operator \( V_2(\cdot) \) is \( L^p \)-stable if whenever the input belongs to \( L^p \), the resulting output belongs to \( L^p \). Moreover, the norm of the output is not larger than \( \alpha_2 \) times the norm of the input plus the offset constant \( \beta_2 \).

Proof of the nonlinear stability proposition

Define the closed-loop mapping \( A(e_1,e_2) \) by:

\[
e_2 = e_2 + e_2 \quad (11)
\]

For each finite \( T \), inequality b is true.

\[
\|e_2_T\|_p \leq \|e_1_T\|_p + \|e_2_T\|_p \quad \forall T < \infty
\]

Figure B1: Simplified Block Diagram of the System in Figure 5
There are two elements in the feedback loop; \( G_2H_2S_{1}^{-1} \) and \( S_{2}S_{1}^{-1} \). \( S_{2}S_{1}^{-1} \) shows the natural force feedback while \( G_2H_2S_{1}^{-1} \) represents the controlled force feedback in the system. The objective is to use Nyquist Criterion (5) to arrive at the sufficient condition for stability of the system when \( H_2 = 0 \). The following conditions are regarded:

1) The closed loop system in Figure B1 is stable if \( H_2 = 0 \). This condition simply states the stability of two robot manipulators. (Figure 2 shows this configuration.)

2) \( H_2 \) is chosen as a stable linear transfer function matrix. Therefore the augmented loop transfer function \((G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1})\) has the same number of unstable poles that \( S_{2}S_{1}^{-1} \) has. Note that in many cases \( S_{2}S_{1}^{-1} \) is a stable system.

3) Number of poles on \( j\omega \) axis for both loop \( S_{2}S_{1}^{-1} \) and \((G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1})\) are equal.

Considering the system in Figure B1 is stable when \( H_2 = 0 \), we plan to find how robust the system is when \( G_2H_2S_{1}^{-1} \) is added to the feedback loop. If the loop transfer function \( S_{2}S_{1}^{-1} \) (without compensator, \( H_2 \)) develops a stable closed-loop system, then we are looking for a condition on \( H_2 \) such that the augmented loop transfer function \((G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1})\) guarantees the stability of the closed-loop system. According to the Nyquist Criterion, the system in Figure B1 remains stable if the anti-clockwise encirclement of the det\((G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1})\) around the center of the s-plane is equal to the number of unstable poles of the loop transfer function \((G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1})\). According to conditions 2 and 3, the loop transfer functions \( S_{2}S_{1}^{-1} \) and \((G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1})\) both have the same number of unstable poles. The closed-loop system when \( H_2 = 0 \) is stable according to condition 1; the encirclements of det\((S_{2}S_{1}^{-1} + I_n)\) is equal to unstable poles of \( S_{2}S_{1}^{-1} \). Since the number of unstable poles of \((G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1})\) and that of \( S_{2}S_{1}^{-1} \) are the same, therefore for stability of the system det\((G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1} + I_n)\) must have the same number of encirclements that det\((S_{2}S_{1}^{-1} + I_n)\) has. A sufficient condition to guarantee the equality of the number of encirclements of det\((G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1} + I_n)\) and that of det\((S_{2}S_{1}^{-1} + I_n)\) is that the det\((G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1} + I_n)\) does not pass through the origin of the s-plane for all possible non-zero but finite values of \( H_2 \), or

\[
\text{det}(G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1} + I_n) = 0 \quad \forall \omega \in (0, \infty) \tag{B1}
\]

If inequality B1 does not hold then there must be a non-zero vector \( z \) such that:

\[
(G_2H_2S_{1}^{-1} + S_{2}S_{1}^{-1} + I_n)z = 0 \quad \forall \omega \in (0, \infty) \tag{B2}
\]

or:

\[
(G_2H_2S_{1}^{-1} + I_n)z = 0 \quad \forall \omega \in (0, \infty) \tag{B3}
\]

A sufficient condition to guarantee that equality B3 will not happen is given by inequality B4.

\[
\sigma_{\text{max}}(G_2H_2S_{1}^{-1}) < \sigma_{\text{max}}(S_{2}S_{1}^{-1} + I_n) \quad \forall \omega \in (0, \infty) \tag{B4}
\]

or a more conservative condition:

\[
\sigma_{\text{max}}(H_2) < \frac{1}{\sigma_{\text{max}}((S_{1}^{-1}S_{2})^{-1}G_2)} \quad \forall \omega \in (0, \infty) \tag{B5}
\]

Note that \((S_{1}^{-1}S_{2})^{-1}G_2\) is the transfer function matrix that maps \( e_2 \) to the contact force, \( f_2 \) when \( e = 0 \). Figure 5 shows the closed-loop system. According to the result of the proposition, \( H_2 \) must be chosen such that the size of \( H_2 \) is smaller than the reciprocal of the size of the forward loop transfer function, \((S_{1}^{-1}S_{2})^{-1}G_2\).

**Appendix C**

The following inequalities are true when \( p \geq 2 \) and \( H_2 \) and \( V_2 \) are linear operators.

\[ \| H_2 f_2 \|_p \leq \| V_2 \|_p \| e_2 \|_p \tag{C1} \]

\[ \| V_2 e_2 \|_p \leq \mu \| e_2 \|_p \tag{C2} \]

where:

\[ \mu = \sigma_{\text{max}}(Q), \text{ and } Q \text{ is the matrix whose ith entry is given by } \left[ Q_{ij} = \sup_{\omega} \left| (V_2)_{ij} \right| \right. \]

\[ \nu = \sigma_{\text{max}}(R), \text{ and } R \text{ is the matrix whose ith entry is given by } \left[ R_{ij} = \sup_{\omega} \left| (H_2)_{ij} \right| \right. \]

According to the stability condition, to guarantee the closed loop stability \( \mu < \frac{1}{\nu} \) or:

\[ \nu < \frac{1}{\mu} \tag{C3} \]

Note that the following are true:

\[ \sigma_{\text{max}}(V_2) < \mu \quad \forall \omega \in (0, \infty) \tag{C4} \]

\[ \sigma_{\text{max}}(H_2) < \nu \quad \forall \omega \in (0, \infty) \tag{C5} \]

Substituting C4 and C5 into inequality C3 which guarantees the stability of the system, the following inequality is obtained:

\[ \sigma_{\text{max}}(H_2) < \frac{1}{\sigma_{\text{max}}(V_2)} \quad \forall \omega \in (0, \infty) \tag{C6} \]

\[ \sigma_{\text{max}}(H_2) < \frac{1}{\sigma_{\text{max}}((S_{1}^{-1}S_{2})^{-1}G_2)} \quad \forall \omega \in (0, \infty) \tag{C7} \]

Inequality C7 is identical to inequality 26. This shows that the linear stability condition by the multivariable Nyquist Criterion is a subset of the general condition given by the Small Gain Theorem.

**References**