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AN APPROACH TO LOOP TRANSFER RECOVERY USING EIGENSTRUCTURE ASSIGNMENT

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1. Abstract

One method of model-based compensator design for linear multivariable systems consists of state-feedback design and observer design [1]. A key step in recent work in multivariable synthesis involves selecting an observer gain so the final loop-transfer function is the same as the state-feedback loop-transfer function [4], [6]. This is called Loop-Transfer Recovery (LTR). This paper shows how identification of the eigenstructure of the compensators that achieve LTR makes possible a design procedure for observer gain [15]. This procedure is based on the eigenstructure assignment of the observers. The sufficient condition for LTR and the stability of the closed-loop system is that the plant be minimum-phase. The limitation of this method might arise when the plant has multiple transmission zeros.

2. Introduction

Historically, the LTR method is the consequence of attempts by Doyle and Stein to improve the robustness of linear quadratic gaussian (LQG) regulators [5], [4]. However, the method has more general applications than to the robustness of the LQG regulators [6]. In their seminal work, Doyle and Stein address the problem of finding the steady-state observer gain that assures the recovery of the loop transfer function resulting from full state feedback. First, they demonstrate a key lemma that gives a sufficient condition for the steady-state observer gain such that LTR takes place. To compute the gain, they show that the infinite time-horizon Kalman filter formalism with "small" white measurement-noise covariance yields an observer gain that satisfies the sufficient condition for loop transfer recovery. In this paper, we present a method for computing observer gain that obviates the need for Kalman filter formalism. The goal of this paper is to analyze the eigenstructure properties of the LTR method for the general class of feedback control systems that use model-based compensators. After examining the eigenstructure of LTR, a design methodology for LTR via eigenstructure assignment will be given.

Nomenclature

A, B& C .... plant parameters
d_i&d_o .... input and output disturbances
$x(t)$,$u(t)$&$y(t)$ ... states, input and output of the system
$\hat{x}(t)$&$\hat{y}(t)$ .... states and output of the observer
σ_i .... transmission zeros of (A,B,G)
G .... state-feedback gain
K(s) .... transfer function matrix of the compensator
ρ .... positive scalar
v_i^T .... left eigenvector of (A-HC)
u_i .... right eigenvector of (A-BG)
W .... square non-singular m×m matrix
z_i^T......... zero direction of the transmission zero
$-w_i^T$ .... input direction of the transmission zero
l .... maximum number of the finite transmission zeros
x_i^T .... left eigenvector of (A-BG-HC)
$\Phi_{ol}(s)$ ..... open-loop characteristic equation of the plant
$\Phi_{cl}(s)$ ..... closed-loop characteristic equation of the observer
n .... order of the system
m .... rank of matrices B and C
P(s) .... precompensator

3. Background

We will deal with the standard feedback configuration shown in Figure 3-1, which consists of: plant model $G_p(s)$; compensator K(s), forced by command r(t); measurement noise n(t); and the disturbances $d_i(t)$ and $d_o(t)$. The precompensator, P(s), is used to filter the input for command following.

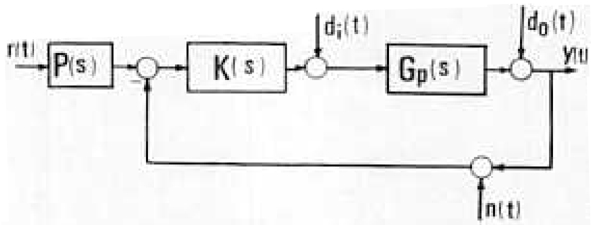

Figure 3-1: Standard Closed-Loop System

Throughout this paper, we assume that the plant can be described by equations 1 and 2.

$$\dot{x}(t) = A\,x(t) + B\,u(t) + B\,d_i(t) \tag{1}$$

$$y(t) = C\,x(t) + d_o(t) + n(t) \tag{2}$$

where:
$x(t) \in R^n$, $u(t), y(t), d_i(t), d_o(t)$ and $n(t) \in R^m$
[A, B] is a stabilizable (controllable) pair
[A, C] is a detectable (observable) pair
rank (B) = *rank* (C) = m

Once we specify the plant model, $G_P(s)$, we must find $K(s)$ so that: 1) the nominal feedback design, $y(s) = G_P(s)[I_{mm} + K(s)G_P(s)]^{-1} d_i(s)$, is stable; 2) the perturbed system in the presence of bounded unstructured uncertainties is stable; 3) application-dependent design specifications are achieved. The design specifications can be expressed as frequency-dependent constraints on the loop transfer function, $K(s)G_P(s)$. The standard practice is to shape the loop transfer function, $K(s)G_P(s)$, so it does not violate the frequency-dependent constraints [4]. The loop-shaping problem can be considered to be a design trade-off among performance objectives, stability in the face of unstructured uncertainties [13, 23], and performance limitations imposed by the gain/phase relationship. Here we assume that $n(t)$ is a noise signal that operates over a frequency range beyond the frequency range of $r(t)$, $d_i(t)$ and $d_o(t)$. We also use a precompensator, $P(s)$, to shape the input for command following. Therefore, the performance objectives are considered as only input disturbance rejection over a bounded frequency range. The design specifications may be frequency-dependent constraints on $G_P(s)K(s)$, which is the loop transfer function broken at the output of the plant, rather than on $K(s)G_P(s)$, which is the loop transfer function broken at the input to the plant. Applying the design specifications to $G_p(s)K(s)$ implies rejection of output disturbances. Since Doyle and Stein first applied LTR to the loop transfer function, $K(s)G_P(s)$, for consistency and continuity we will also assume throughout this article that all design specifications apply to $K(s)G_P(s)$.

One method of designing $K(s)$ consists of two stages. The first stage concerns state-feedback design. A state-feedback gain, G, is designed so that the loop transfer function, $G(sI_{nn}-A)^{-1}B$, which is shown in Figure 3-2, meets the frequency-dependent design specifications and satisfies equation 3 to guarantee stability.

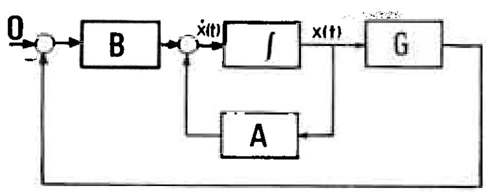

Figure 3-2: State-Feedback Configuration

$$\lambda_i I_{\ } \; - A + B\,G\;)\;u_i = 0, \qquad i = 1, 2, \tag{3}$$

$$\text{real}\;(\;\lambda_{\ } \;<0 \qquad u_i \neq 0_{\ }$$

λ_i is the closed-loop state-feedback eigenvalue, while u_i is the $n \times 1$ right closed-loop eigenvector of the system. Controllability of [A,B] guarantees the existence of G in equation 3. At this stage, one can determine whether or not state-feedback design can meet the design specifications. In this paper, we assume that G is selected so that equation 3 is satisfied and the loop transfer function, $G(sI_{nn}-A)^{-1}B$, which is shown in Figure 3-2 meets the desired frequency-domain design specification. In the second stage of the compensator design, an observer is designed to make the first stage realizable [14, 25]. The observer design is not involved in meeting the specifications for the loop transfer function since all design specifications have been met by the state feedback gain, G.
The observer has the structure of the Kalman filter. Combining the state-feedback and observer designs (Figure 3-3) yields the unique compensator transfer-function matrix

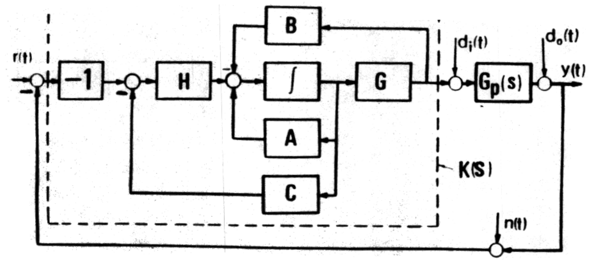

Figure 3-3: Closed-Loop System

given by equation 4.

$$K(s) = G\;(\;s \qquad A + B\,G + H\,C\;)^{-1} H \tag{4}$$

The idea behind observer design is to find the steady-state gain, H, such that the loop transfer function, $K(s)G_P(s)$, in Figure 3-1 maintains the same loop shape (for a bounded frequency range) that $G(sI_{nn}-A)^{-1}B$ achieved via state-feedback design in the first stage. A technique for designing H to meet this criterion was offered by Doyle and Stein [4]. Since by this method, $K(s)G_P(s)$ preserves the loop-shape achieved by $G(sI_{nn}-A)^{-1}B$, the final design in Figure 3-1 meets the specifications that were already met by state-feedback design. (The title "loop transfer recovery" comes from this idea.) For stability of the observer, equation 5 must also be satisfied.

$$v_i^T(\;\mu_i\,I_{nn} - A + H\,C \quad = 0_{\ }^T, \quad i = 1, 2, \tag{5}$$

$$\text{real}\;(\;\mu_i \;<0 \qquad v_i^T \neq 0_{\ }$$

μ_i and v_i^T are the observer eigenvalue and left eigenvector, respectively. Observability of [A,C] guarantees the existence of H in equation 5. The following lemma, which is proved by Doyle and Stein [4], is central to the design of H:

If H *is chosen such that limit* 6 *is true as scalar approaches infinity for any non-singular* $m \times m$ W*-matrix* ,

$$\frac{H(\rho)}{\rho} \rightarrow BW \tag{6}$$

then $K(s)$, *as given by equation* 4, *approaches pointwise toward expression* 7:

$$G\,(sI_{nn}-A)^{-1}\,B\;[\;C\,(sI_{nn}-A)^{-1}\,B\;]^{-1}, \tag{7}$$

and since $\qquad G_P(s) = C\,(sI_{nn}-A)^{-1}\,B \qquad$ (8)

then $K(s)\,G_P(s)$ *will approach* $G\,(sI_{nn}-A)^{-1}\,B$ *pointwise.*

The procedure requires only that H be stabilizing and have the asymptotic characteristic of equation 6. Doyle and Stein suggested one way to meet this requirement: a steady-state Kalman filter gain [12] with very small measurement-noise covariance. Now suppose we choose H with the following structure:

$$H = \rho\,B\,W \tag{9}$$

where W is any non-singular $m \times m$ matrix and ρ is a scalar.

It can be shown (by the definition of the limit) that the structure of H chosen in equation 9 satisfies the limit in equation 6 as ρ approaches infinity. In other words, as ρ approaches infinity, '$H \to \rho BW$' results in '$H/\rho \to BW$'. (The reverse is not true.) Since the structure of H given in equation 9 satisfies the limit in 6, then if H is chosen to be ρBW, $K(s)G_p(s)$ will approach $G(sI_{nn}-A)^{-1}B$ pointwise, as ρ approaches infinity. Note that the structure of H given by equation 9 does not necessarily yield a stable observer. We choose H to be ρBW throughout this paper. The asymptotic finite eigenstructures of both forms given by 9 and 6 are the same, while the asymptotic infinite eigenstructures are usually different. The form in equation 9 usually yields an unstable infinite eigenstructure.

Although this paper is not an exposition of the properties of the transmission zeros of a plant, before stating the theorem, we will remind readers of some definitions and concepts about this matter. (For more information and properties of the transmission zeros, see references [22, 3, 10].) The transmission zeros of a square plant are defined to be the set of complex numbers s_i that satisfy inequality 10.

$$rank \begin{bmatrix} s_iI_{nn} - A & B \\ C & 0_{mm} \end{bmatrix} < n + m \qquad (10)$$

The necessary and sufficient condition for the truth of inequality 10 is given by equation 11.

$$det \begin{bmatrix} s_iI_{nn} - A & B \\ C & 0_{mm} \end{bmatrix} = 0 \qquad (11)$$

Equation 11 yields l finite transmission zeros ($l \leq n-m$). The remaining (n-l) transmission zeros are at infinity. For each finite transmission zero, there is one non-zero left null-vector $[z_i^T \quad -w_i^T]$ (for i=1,2,...,l) such that:

$$[z_i^T \quad -w_i^T] \begin{bmatrix} s_iI_{nn} - A & B \\ C & 0_{mm} \end{bmatrix} = 0_{n+m}^T \qquad (12)$$

$$\text{where: } [z_i^T \quad -w_i^T] \neq 0_{n+m}^T$$

w_i^T is an $m \times 1$ vector, and z_i^T is an $n \times 1$ vector. z_i^T is called left zero direction of the transmission zeros of the plant. If the left eigenvector and left zero direction associated with a pair of equal-valued eigenvalue and transmission zero are equal, then s_i is an uncontrollable mode of the system. These transmission zeros are called "input decoupling zeros" [17]. Similarly, if the right eigenvector and the right zero direction associated with an equal-valued eigenvalue and transmission zero are equal, then s_i is an unobservable mode of the system. These transmission zeros are called "output decoupling zeros." A similar definition for the transmission zeros of a square plant is given by reference [12]; all complex numbers that are roots of $\Psi(s)$ in the equation:

$$det\ G_p(s) = \frac{\Psi(s)}{\Phi_{ol}(s)} \qquad (13)$$

are transmission zeros of the plant. $\Phi_{ol}(s)$ is the nth-order open-loop characteristic equation. The maximum order of $\Psi(s)$ is l. All transmission zeros of the plant, including the ones that are equal to the eigenvalues of the plant (which may even be the input-decoupling and/or output-decoupling zeros of the system), are roots of $\Psi(s)$ and also satisfy inequality 10 and equation 11. The equality of equations 13 and 11 can be shown by careful use of Schur's equality [8].

4. Asymptotic Eigenstructure Properties of the LTR Method

We will now explore some eigenstructure properties for LTR when the observer gain satisfies equation 9. Knowing the eigenstructure properties of the compensator, we will develop a method for designing H via eigenstructure assignment of the observer. The following theorem gives the eigenstructure properties of the observer when H is chosen according to equation 9. Part 1 of the theorem is proved differently in reference [2], and can also be considered to be a special result of the multivariable root locus given by references [18, 24, 11, 10]. The second part of the theorem is the result we will use in the design process.

Theorem

Consider the square linear observer in Figure 4-1

$$\dot{\hat{x}}(t) = A\hat{x}(t) + H e(t) + B u(t) \qquad (14)$$

$$e(t) = -C\hat{x}(t) + y(t) \qquad (15)$$

$$\hat{x}(t) \in R^n \qquad u(t) \text{ and } y(t) \in R^m$$

$$\text{with } rank\,(B) = rank\,(C) = m$$

Then if H is chosen so that.

$$H = \rho B W \qquad (16)$$

where W is any non-singular square matrix and ρ is a scalar approaching ∞, then the following statements are true:

1) *The finite closed-loop eigenvalues of (A-HC), μ_i, approach finite transmission zeros of the plant, s_i. If the linear plant [A,B,C] has l finite transmission zeros, ($l \leq n-m$), then (A-HC) will have l finite eigenvalues. The remaining closed-loop eigenvalues approach infinity at any angle.*

2) *The left closed-loop eigenvector v_i^T (i=1,2,...,l) associated with the finite closed-loop eigenvalue μ_i approaches z_i^T, which satisfies equation 17.*

$$[z_i^T \quad -w_i^T] \begin{bmatrix} s_iI_{nn}-A & B \\ C & 0_{mm} \end{bmatrix} = 0_{n+m}^T \qquad (17)$$

$$[z_i^T \quad -w_i^T] \neq 0_{n+m}$$

w_i^T is an $m \times 1$ vector and z_i^T is an $n \times 1$ vector. *If s_i is not equal to any eigenvalue of A, then z_i^T can be computed from equation 17, and the following expression for v_i^T (i=1,2,...,l) can be obtained:*

$$v_i^T = w_i^T [\, C\,(\, s_i\, I_{nn} - A\,)^{-1}\,] \qquad (18)$$

where w_i^T (i=1,2,...,l) can be calculated from equation 19.

$$w_i^T [C (s_i I_{nn} - A)^{-1} B] = 0_m^T \tag{19}$$

where: $w_i^T \neq 0_m^T$

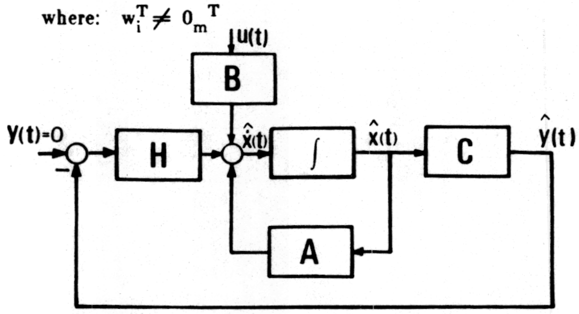

Figure 4-1: Closed-Loop Observer

Interpretation. This theorem identifies the asymptotic locations of finite closed-loop eigenvalues and left eigenvectors of the observer. As ρ approaches a large number, l (for $l \leq n-m$) closed-loop eigenvalues will approach finite transmission zeros of the plant, and (n-l) closed-loop eigenvalues will approach infinity at any angle. Since conventional practice in complex variable work is to regard a function as having an equal number of poles and zeros when the zeros at infinity are included, one can claim that all closed-loop eigenvalues approach the transmission zeros of the plant. Equation 17 states that $[\ v_i^T \quad -w_i^T\ ]$ is confined in the left null space of the given matrix in equation 17 as ρ approaches infinity. In other words, the left null space of the matrix given in equation 17 assigns a subspace for limiting location of $[\ v_i^T \quad -w_i^T\ ]$ when ρ approaches infinity. If s_i is not equal to any eigenvalues of A, the limiting location of v_i^T can be interpreted differently. Equation 18 states that the left eigenvector v_i^T is confined to a sub-space spanned by the rows of $[C(s_i I_{nn}-A)^{-1}]$ if s_i is not equal to any eigenvalues of A. This sub-space is of dimension equal to the rank of C. Therefore, the number of independent output variables determines how large the sub-space corresponding to the left closed-loop eigenvector can be. The orientation of each sub-space associated with each left closed-loop eigenvector v_i^T depends on the open-loop dynamics of the system [A,C] and the closed-loop observer eigenvalue μ_i. Construction of the left closed-loop eigenvectors in their allowable m-dimensional sub-space in C^n is the exact freedom that is offered by observer design beyond pole placement [9, 19, 20, 7]. The second part of the theorem identifies the asymptotic m-dimensional sub-space in R^n that confines the left closed-loop eigenvector v_i^T. The choice of w_i^T in equation 18 allows the designer to construct each n-dimensional left closed-loop eigenvector in its allowable m-dimensional sub-space. As ρ approaches a large number, then w_i^T approaches the left null vector of $G_p(s_i)$ in equation 19; consequently, each left closed-loop eigenvector v_i^T approaches a final value in its allowable sub-space given by expression 18.

Proof:

Part 1: H is chosen according to equation 16. The block diagram of the closed-loop observer is shown in Figure 4-2. The loop transfer function at the plant output is given by expression 20.

$$C (s \qquad A)^{-1} \rho B W \tag{20}$$

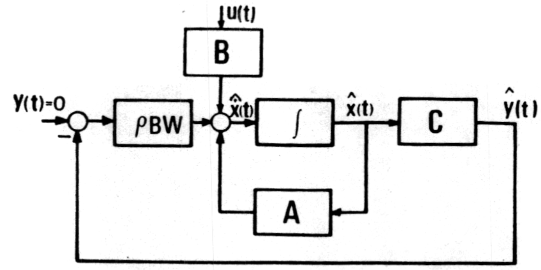

Figure 4-2: Closed-Loop Observer Configuration

Equation 21 relates the open-loop and closed-loop characteristic equations [16, 21].

$$det\ [\ I_{nn} + C\ (sI_{nn}-A)^{-1}\ \rho\ B\ W \ = \frac{\Phi_{cl}(s)}{\Phi_{ol}(s)} \tag{21}$$

where:

$\Phi_{cl}(s)$ = closed-loop characteristic equation of the system in Figure 4-2.

$\Phi_{ol}(s)$ = open-loop characteristic equation of the system in Figure 4-2.

From matrix theory, equality 22 is true [8].

$$det\ [\ I_{mm} + C\ (sI_{nn}-A)^{-1}\ \rho\ B\ W\ ] = I_{mm} +$$

$$trace[C(sI_{nn}-A)^{-1}\ \rho BW] + \ \dots\ + det[C(sI_{nn}-A)^{-1}\rho BW] \tag{22}$$

As ρ approaches ∞, the last term of equation 22 grows faster than the other terms. Therefore, approximation 23 is true.

$$det[\ I_{mm} + C(sI_{nn}-A)^{-1}\ \rho BW] \approx det[C(sI_{nn}-A)^{-1}\ \rho BW] \tag{23}$$

Considering approximation 23, equation 21 can be written as:

$$det\ [\ C\ (sI_{nn}-A)^{-1}\ \rho\ B\ W\ ] \approx \frac{\Phi_{cl}(s)}{\Phi_{ol}(s)} \tag{24}$$

or equivalently:

$$det\ [G_P(s)]\ det\ [\rho\ W] \approx \frac{\Phi_{cl}(s)}{\Phi_{ol}(s)} \tag{25}$$

Since $det\ [\rho\ W] \neq 0$, comparing equations 13 and 25 shows that the roots of $\Psi(s)$ and $\Phi_{cl}(s)$ are the same. In other words, $\Phi_{cl}(s)$ produces all the transmission zeros of the plant, including the ones that are equal to the eigenvalues of A, which can even be decoupling zeros.

Part 2: When H approaches its asymptotic value, the eigenvalues of (A-HC) can no longer be moved via matrix C. This is true because the eigenvalues of (A-HC) are at their limiting locations (i.e., transmission zeros of the plant). Therefore, [(A-HC), B, C] must have unobservable or uncontrollable modes. Since [(A-HC), C] is an observable pair and H is expressed as ρBW, [(A-HC), B] must be an uncontrollable pair. Since [(A-HC), B] is an uncontrollable pair, equations 26 and 27 are true [17].

$$v_i^T (\ \mu_i\ I_{nn} - A + H\ C \ = 0_{\cdot}^T \qquad = 1, 2, \dots, l \tag{26}$$

$$v_i^T B \ = 0_m^T \tag{27}$$

μ_i is the closed-loop observer eigenvalue, and v_i^T is the corresponding left eigenvector. Equation 27 states that the left closed-loop eigenvector v_i^T from equation 26 is in the left null space of B and cannot be affected by the input. Each closed-loop eigenvector v_i^T (for i=1,2,...,l) can be expressed by equation 28.

$$v_i^T(\mu_i I_{nn} - A) - w_i^T C = 0_n^T \tag{28}$$

$$\text{where:} \quad w_i^T = -v_i^T H \tag{29}$$

Combining equation 28 and equation 27 yields equation 30. (Note that $s_i = \mu_i$.)

$$\begin{bmatrix} v_i^T & -w_i^T \end{bmatrix} \begin{bmatrix} s_i I_{nn} - A & B \\ C & 0_{mm} \end{bmatrix} = 0_{n+m}^T \tag{30}$$

$$\text{where: } [\, v_i^T \quad -w_i^T \,] \neq 0_{n+m}^T \quad \text{for } i=1,2,...,l$$

If s_i is not equal to any eigenvalue of A, then from equation 30 we can find an expression for the left closed-loop eigenvector of A:

$$v_i^T = w_i^T C\,(s_i I_{nn} - A)^{-1} \quad i = 1, 2, ..., l \tag{31}$$

where w_i^T can be computed from equation 32

$$w_i^T [\, C\,(s_i I_{nn} - A)^{-1} B \,] = 0_m^T, \quad i=1,2,...,l \tag{32}$$

$$\text{where: } w_i^T \neq 0_m^T$$

Equation 31 shows that the left eigenvectors achievable for the closed-loop observer are confined to the m-dimensional sub-spaces determined by their associated eigenvalues and open-loop dynamics [A, C].

Comment: As ρ approaches ∞, the l eigenvalues of (A-HC) cancel out the l finite transmission zeros of the plant. A cancellation of an equal-valued closed-loop eigenvalue of the system with a transmission zero happens if the left closed-loop eigenvector of the system is equal to the left zero direction, z_i^T, associated with the transmission zero in equation 17. By cancelling we mean they will not appear as poles in the closed-loop transfer function matrix, $C[sI_{nn}-A+HC]^{-1}B$. The transmission zeros of [A, B, C] are the same as those of [(A-HC), B, C], because transmission zeros do not change under feedback. As ρ approaches infinity, the transmission zeros of [(A-HC), B, C] turn into input decoupling zeros, because the system of [(A-HC), B, C] is not controllable at these modes [17].

Corollary 1: *The finite transmission zeros of* K(s) *are the same as the finite transmission zeros of* $G(sI_{nn}-A)^{-1}B$.

Proof: The transmission zeros of $G(sI_{nn}-A)^{-1}B$ are the complex values σ_i that satisfy the following inequality :

$$rank \begin{bmatrix} \sigma_i I_{nn} & A & B \\ G & & 0_{mm} \end{bmatrix} < n + m \tag{33}$$

Post-multiplying the matrix in inequality 33 by the non-singular matrix:

$$\begin{bmatrix} I_{nn} & 0_{nm} \\ G + \rho W C & \rho W \end{bmatrix} \tag{34}$$

will result in inequality 35 for the transmission zeros of $G(sI_{nn}-A)^{-1}B$:

$$rank \begin{bmatrix} \sigma_i I_{nn} - A+BG+\rho BWC & \rho BW \\ G & 0_{mm} \end{bmatrix} < n+m \tag{35}$$

Substituting H for (ρBW) in inequality 35 results in inequality 36.

$$rank \begin{bmatrix} \sigma_i I_{nn}-A+BG+HC & H \\ G & 0_{mm} \end{bmatrix} < n+m \tag{36}$$

The complex number σ_i that satisfies inequality 36 is a transmission zero of K(s) as given by equation 4. Therefore, K(s) and $G(sI_{nn}-A)^{-1}B$ have equal transmission zeros. If $G(sI_{nn}-A)^{-1}B$ does not have any finite transmission zeros, then K(s) will not have any finite transmission zeros.

Corollary 2: *If ρ approaches ∞, then all the eigenvalues of the compensator* K(s) *will approach the transmission zeros (including the ones at infinity) of the plant, and the left eigenvectors of* (A-BG-HC), x_i^T, *will approach* z_i^T, *where* z_i^T *and* s_i (i=1,2,...,l) *satisfy equation* 37.

$$\begin{bmatrix} z_i^T & -w_i^T \end{bmatrix} \begin{bmatrix} s_i I_{nn} - A & B \\ C & 0_{mm} \end{bmatrix} = 0_{n+m}^T \tag{37}$$

$$[\, z_i^T \quad -w_i^T \,] \neq 0_{m+n}^T$$

In other words, the eigenvalues of the compensator cancel out the transmission zeros of the plant.

Proof: The transmission zeros of the plant are the set of complex numbers s_i that satisfy inequality 37. Post-multiplying the matrix in equation 37 by the non-singular matrix:

$$\begin{bmatrix} I_{nn} & 0_{nm} \\ G & I_{mm} \end{bmatrix} \tag{38}$$

will yield the following equation, which can then be solved to find the finite transmission zeros of the plant:

$$\begin{bmatrix} z_i^T & -w_i^T \end{bmatrix} \begin{bmatrix} s_i I_{nn} - A+BG & B \\ C & 0_{mm} \end{bmatrix} = 0_{n+m}^T \tag{39}$$

$$[\, z_i^T \quad -w_i^T \,] \neq 0_{n+m}^T \quad \text{for } i=1, 2, ..., l.$$

We apply the result of the theorem to system [(A-BG), B, C]. According to part 1 of the theorem, if H=ρBW, then as ρ approaches ∞, the eigenvalues of (A-BG-HC) will approach the transmission zeros of [(A-BG), B, C] computed from equations 39. These are also the transmission zeros of the plant given by equation 37.

According to part 2 of the theorem, the left closed-loop eigenvectors x_i^T of the compensator given by equation 40:

$$x_i^T(\mu_i I_{nn}-A+BG+HC) = 0_n^T, \quad i=1,2,...,l \tag{40}$$

$$x_i^T \neq 0_n^T$$

approach z_i^T given by equation 39 or equation 37.

5. Comments

1) According to corollary 2, as ρ approaches ∞, the eigenvalues of K(s) will cancel out the transmission zeros of the plant. According to corollary 1, as ρ approaches ∞, the transmission zeros of K(s) will approach the transmission zeros of $G(sI_{nn}\text{-}A)^{-1}B$. Since the number of transmission zeros of two cascaded systems (K(s) and $G_P(s)$) is the sum of the number of transmission zeros of both systems, the transmission zeros of $K(s)G_P(s)$ are the same as the transmission zeros of $G(sI_{nn}\text{-}A)^{-1}B$. Similar arguments can be given for the poles of $K(s)G_P(s)$. The poles of K(s) cancel out the transmission zeros of the plant; therefore, the poles of $K(s)G_P(s)$ will be the same as poles of $G(sI_{nn}\text{-}A)^{-1}B$. This argument does not prove the equality of $G(sI_{nn}\text{-}A)^{-1}B$ and $K(s)G_P(s)$ as ρ approaches ∞. Proof of the pointwise equality of $K(s)G_P(s)$ and $G(sI_{nn}\text{-}A)^{-1}B$ is best shown by Doyle and Stein in [4]. The above comment concerning pole-zero cancellation explains the eigenstructure mechanism for LTR. Since pole placement and eigenvector construction in the allowable sub-space prescribes a unique value for H, we plan to design the observer gain for the LTR via pole placement and left eigenvector construction .

2) The asymptotic finite eigenstructure for H in both equations 6 and 9 are the same, but the asymptotic infinite eigenstructures are usually different. The form of H given by equation 9 is rarely stabilizing. Since both forms guarantee the pointwise approach of $K(s)G_p(s)$ to $G(sI_{nn}\text{-}A)B$, it can be deduced that the pointwise approach of $K(s)G_p(s)$ to $G(sI_{nn}\text{-}A)B$ occurs whenever the asymptotic finite eigenstructure is the same as that given by the theorem. Hence, combining any such finite eigenstructure with any stable infinite eigenstructure will result in the approach of $K(s)G_p(s)$ to $G(sI_{nn}\text{-}A)B$ in a stable sense.

3) Difficulty in using LTR will arise if the plant has some right half-plane zeros (non-minimum phase plant). In our proposed procedure for LTR, one should place the eigenvalues of (A-HC) at the transmission zeros of the plant. If the plant is non-minimum phase, one would place some eigenvalues of (A-HC) on the right half-plane. The closed-loop system will not be stable if any eigenvalues of (A-HC) are on the right half-plane. According to the separation theorem, the eigenvalues of (A-HC) are also the eigenvalues of the closed-loop system. Therefore, the sufficient condition for LTR and the stability of the closed-loop system is that the plant be minimum-phase. If the plant is non-minimum phase, one should consider the mirror images of the right half-plane zeros as target locations for eigenvalues of (A-HC). In such cases, loop transfer recovery is not guaranteed, but the closed-loop system will be stable.

6. Design Method

For observer design, we place l finite eigenvalues of (A-HC) at finite transmission zeros of the plant. The left closed-loop eigenvector v_i^T associated with the finite modes must be constructed such that $[v_i^T \;\; -w_i^T]$ is in the left null space of the matrix given by equation 17. The remaining (n-l) closed-loop eigenvalues should be placed far in the left half-plane. Note that the farther the (n-l) infinite eigenvalues of (A-HC) are located from the imaginary axis, the closer $K(s)G_p(s)$ will be to $G(sI_{nn}\text{-}A)B$. This is shown in the example. The left closed-loop eigenvectors associated with the infinite modes can be computed via equation 41.

$$v_i^T = w_i^T C \; (\, \mu_i \, I_{nn} \quad A \,)^{-1} \qquad i = l+1, \, l+2, \qquad (41)$$

$$\text{where} \quad w_i^T = - \, v_i^T H \qquad (42)$$

The following steps will lead a designer toward observer design for the recovery procedure:

1) Use equation 17 to compute the l target locations of the complex finite eigenvalues of the observer, s_i, and l left null vectors of $[\, z_i^T \;\; -w_i^T \,]$. μ_i must be selected to be equal to s_i. The left closed-loop eigenvector of the observer, v_i^T, must be selected to be equal to z_i^T. If s_i is not equal to any eigenvalue of A, use equations 18 and 19 to compute the l left closed-loop eigenvectors v_i^T and w_i^T . w_i^T identifies the location of the left closed-loop eigenvector in its allowable sub-space. This step terminates the construction of finite eigenstructure of the observer.

2) Place the remaining (n-l) eigenvalues of (A-HC) at locations farther than the finite transmission zeros of the plant. Use equation 41 to achieve (n-l) values for v_i^T. The w_i^T for infinite modes are arbitrary and have little importance, because their corresponding eigenvalues are selected far in the left half complex plane.

3) Since

$$v_i^T H = - \, w_i^T \qquad i = 1, \, 2, \qquad n \qquad (43)$$

then:

$$\begin{bmatrix} v_1^T \\ v_2^T \\ \cdot \\ v_n^T \end{bmatrix} H = - \begin{bmatrix} w_1^T \\ w_2^T \\ \cdot \\ w_n^T \end{bmatrix} \qquad (44)$$

Use equation 45 to compute H.

$$H = - \begin{bmatrix} v_1^T \\ v_2^T \\ \cdot \\ v_n^T \end{bmatrix}^{-1} \begin{bmatrix} w_1^T \\ w_2^T \\ \cdot \\ w_n^T \end{bmatrix} \qquad (45)$$

The independence of the n left closed-loop eigenvectors v_i^T is a necessary condition to use eigenstructure assignment for LTR. If the left closed-loop eigenvectors are not independent, our approach fails and one must use Doyle and Stein's approach to recover the loop transfer function. The dependency of the left eigenvectors might arise if multiple finite transmission zeros result in equation 17. If degeneracy of the matrix in equation 17 is equal to the multiplicity of a transmission zero, the existence of n independent finite left closed-loop eigenvectors is guaranteed.

7. Example

Consider the following example:

$$A = \begin{bmatrix} 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 1. \\ 0. & 0. & 0. & 0. \\ 0. & 0 & 0 & 0. \end{bmatrix} \qquad B = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \\ 76.0 & -105.0 \\ -105.0 & 280.0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0. & 1 & 0. \\ 0 & 1. & 0 & 4. \end{bmatrix}$$

Suppose we are given G such that the closed-loop poles are at -19.35, -1.76, -5.57 and -6.12 :

$$G = \begin{bmatrix} 4.7234 & 3.4265 & 0.9923 & 0.6631 \\ 1.1497 & 0.8579 & 0.2633 & 0.1952 \end{bmatrix}.$$

Using equation 19, the finite transmission zeros s_i and the associated left null-vector directions w_i^T can be computed. μ_1 and μ_2 are selected to be equal to s_1 and s_2.

$$\mu_1 = -1, \ \mu_2 = -.25, \ w_1^T = [1 \quad 0], \ w_2^T = [0 \quad 1] \ .$$

Using equation 18, the left closed-loop eigenvector associated with the finite modes can be computed:

$$v_1^T = [\ -1.00 \quad 0.00 \quad 0.00 \quad 0.00\ ]$$

$$v_2^T = [\ 0.00 \quad -4.00 \quad 0.00 \quad 0.00\ ] \ .$$

We place the other two eigenvalues of (A-HC) in the left half-plane as far as possible. The directions of $w_3{}^T$ and $w_4{}^T$ do not matter because the associated eigenvalues are far away. Figure 7-1 shows that the farther away from the imaginary axis the two infinite eigenvalues of (A-HC) are, the closer $K(s)G_p(s)$ will be to $G(sI_{nn}-A)^{-1}B$. Assuming :

$$\mu_3 = -30, \ \mu_4 = -36, \ w_3^T = [1 \quad 0], \ w_4^T = [0 \quad 1]$$

and using equation 41, the left eigenvectors associated with infinite modes can be computed:

$$v_3^T = [\ -0.0333 \quad 0.0000 \quad -0.0322 \quad 0.0000$$

$$v_4^T = [\ 0.0000 \quad -0.0278 \quad 0.0000 \quad -0.1103$$

Using equation 45, H can be computed:

$$H = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 0.2500 \\ 30.0000 & 0.0000 \\ 0.0000 & 0.9000 \end{bmatrix}$$

The finite transmission zeros of $G(sI_{nn}-A)^{-1}B$ are located at -4.3270 and -1.3675 . Table 7-1 shows that the transmission zeros of K(s) approach the transmission zeros of $G(sI_{nn}-A)^{-1}B$ as μ_3 and μ_4 move farther into the left half complex plane (corollary 1). Table 7-1 also shows that the farther μ_3 and μ_4 are from the imaginary axis, the closer the eigenvalues of K(s) will be to the transmission zeros of the plant (corollary 2).

Closed loop eigenvalues	Transmission zeros of K(s)	Eigenvalues of K(s)
μ_1 = - 1 μ_2 = -.25 μ_3 = -10 μ_4 = -12	-2.8097 -1.2662 $-\infty$ $-\infty$	-30.5672 -24.2387 -1.0000 -0.2500
μ_1 = -1 μ_2 = -.25 μ_3 = -30 μ_4 = -36	-3.6734 -1.3311 $-\infty$ $-\infty$	-49.4030 +10.4478i -49.4030 -10.4478i -1.0000 -0.2500
μ_1 = -1 μ_2 = -.25 μ_3 = -90 μ_4 = -108	-4.0855 -1.3551 $-\infty$ $-\infty$	-115.40 + 20.46i -115.40 - 20.46i -1.0000 -0.25

Table 7-1: Poles and Zeros of K(s)

the plant be minimum-phase. The limitation of this method might arise when the plant has multiple finite transmission zeros, and n left independent closed-loop eigenvector cannot be constructed.

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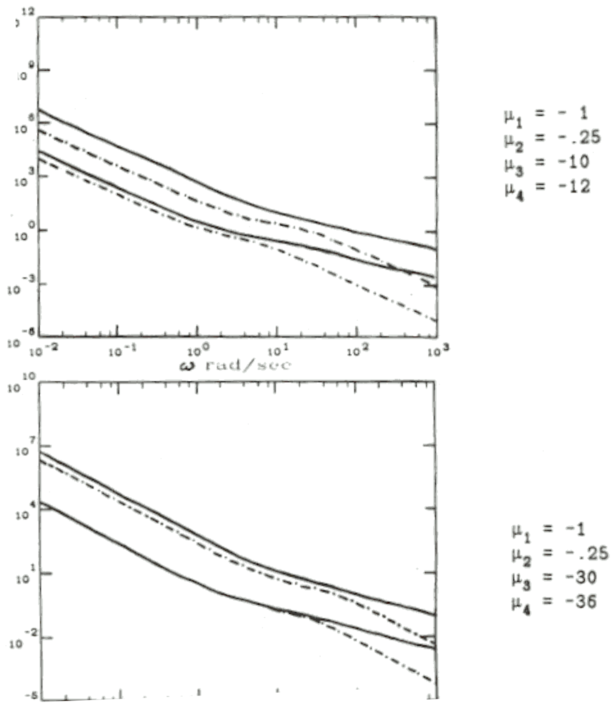

—— Maximum and Minimum Singular Values of $G(sI_n-A)^{-1}B$

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