Compliance control and stability analysis of cooperating robot manipulators
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SUMMARY
The work presented here is the description of the control strategy of two cooperating robots. A two-finger hand is an example of such a system. The control method allows for position control of the contact point by one of the robots while the other robot controls the contact force. The stability analysis of two robot manipulators has been investigated using unstructured models for dynamic behavior of robot manipulators. For the stability of two robots, there must be some initial compliance in either robot. The initial compliance in the robots can be obtained by a non-zero sensitivity function for the tracking controller or a passive compliant element such as an RCC.

KEYWORDS: Compliance; Stability analysis; Cooperating robots; Control strategy.

INTRODUCTION
The paper develops the essential rules in stability analysis of two cooperating robots. We assume the robots initially have some type of independent tracking capabilities. This assumption permits us to extend the control analysis to cover industrial robot manipulators in addition to research robots. The tracking capability allows each robot to follow its individual command independently when it is not constrained by each other. Once the robots come in contact with each other, the contact force between the two robots is fed back to one of the robots to develop compliance.1-4 The compliance in one of the robots allows for control of the contact force, while the other robot governs the position of the contact point. A stability bound has been developed on the size of the force feedback gain to stabilize the closed loop system of both robots. The stability analysis has been investigated using unstructured models for the dynamic behavior of the robot manipulators. This unified approach of modeling robot dynamics is expressed in terms of sensitivity functions as opposed to the Lagrangian approach. It allows us to incorporate the dynamic behavior of all the elements of a robot manipulator (i.e. actuators, sensors and the structural compliance of the links) in addition to the rigid body dynamics.4

DYNAMIC MODEL OF THE ROBOT
In this section, a general approach will be developed to industrial and research robot manipulators having positioning (tracking) controllers. The fact that most industrial manipulators already have some kind of positioning controller is the motivation behind our approach. Also, a number of methodologies exist for the development of robust positioning controllers for direct and non-direct robot manipulators.5

In general, the end-point position of a robot manipulator that has a positioning controller is a dynamic function of its input trajectory vector, e, and the external force, f. Let G and S be two functions that describe the robot end-point position, y, in a global coordinate frame. (f is measured in the global coordinate frame also.)

\[ y = G(e) + S(f) \] (1)

The motion of the robot end-point in response to imposed forces, f, is caused either by structural compliance in the robot or by the compliance of the positioning controller. In a simple example, if a Remote Center Compliance (RCC) with a linear dynamic behavior is installed at the endpoint of the robot, then \( S \) is equal to the reciprocal of stiffness (impedance in the dynamic sense) of the RCC. Robots with tracking controllers are not infinitely stiff in response to external forces (also called disturbances). Even though the positioning controllers of robots are usually designed to follow the trajectory commands and reject disturbances, the robot end-point will move somewhat in response to imposed forces on it. \( S \) is called the sensitivity function and it maps the external forces to the robot end-point position. For a robot with a “good” positioning controller, \( S \) in a mapping with small gain. No assumption on the internal structures of \( G(e) \) and \( S(f) \) are made. Figure 1 shows the nature of the mapping in equation (1).

Figure 2 shows one possible example of internal structure of the model represented by equation (1). The
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transfer functions as general representations of the dynamic behavior of the components of the robot (e.g. servovalves in the hydraulic robots and the gear stiffness in the non-direct drive systems). Throughout this paper we assume the robot dynamic behavior is given by equation (1) where \( G(e) \) and \( S(f) \) can be computed experimentally or analytically from the closed loop block diagram similar to the one given in Figure 2. A robot with good tracking capability has a small gain for \( S \) (rejects all the forces) while a robot with a weak tracking capability has a large gain for \( S \). In fact, an open loop robot—which has the weakest tracking capability—can be modeled with the largest gain on \( S \). If we define an open loop robot as a system with very small feedback gain (\( K_p \) and \( K_v \) in the case of Figure 2) then equation (1)—with a large gain for \( S \)—can be used to model the open loop robots also. Therefore we define \( G(e) \) and \( S(f) \) as stable, nonlinear operators in L_p-space to represent the dynamic behavior of not only the closed loop robots but also the open loop (in the sense of above definition) robots. \( G(e) \) and \( S(f) \) are such that \( \|G(e)\|_p \leq \alpha_G \|e\|_p + \beta_G \) and \( \|S(f)\|_p \leq \alpha_S \|f\|_p + \beta_S \). \( \alpha_s \) is called the gain of operator \( S \).

A similar modeling method can be given for analysis of the linearly treated robots.* The transfer function matrices, \( G \) and \( S \) in equation (2) are defined to describe the dynamic behavior of a linearly treated robot manipulator with positioning controller.

\[
y(j\omega) = G(j\omega)e(j\omega) + S(j\omega)f(j\omega)
\]

In equation (2), \( S \) is called the sensitivity transfer function matrix and it maps the external forces to the end-point position. \( G(j\omega) \) is the closed loop transfer function matrix that maps the input trajectory vector, \( e \), to the robot end-point position, \( y \). For a robot with a “good” positioning controller, within the closed loop bandwidth \( S(j\omega) \) is “small” in the singular value sense, while \( G(j\omega) \) is approximately a unity matrix. We define

*Throughout this paper, for the benefit of clarity, we develop the frequency domain theory for linearly treated robots in parallel with the nonlinear analysis.
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$S^{-1}$ as inverse function of the $S$ function

\[ f = S^{-1}(y - G(e)) \]  

(3)

**DYNAMICS OF TWO ROBOTS**

Suppose two manipulators with dynamic equation (1) are in contact with each other. Equations (4) and (5) represent the entire dynamic behavior of two interacting robots.

\[ y_1 = G_1(e_1) + S_1(f_1) \]  

(4)

\[ f_2 = S_2^{-1}(y_2 - G_2(e_2)) \]  

(5)

where

\[ y_1 = y_2 \text{ and } f_1 = -f_2 \]

Figure 3 shows the block diagram of the interaction of two robots. Note that the blocks in Figure 3 are in general non-linear operators, however, in the linear case one can treat these blocks as transfer function matrices. If all the operators of the block diagram in Figure 3 were transfer function matrices, then the contact force, $f_2$, could be calculated from equation (6).

\[ f_2 = (S_1 + S_2)^{-1}(G_1e_1 - G_2e_2) \]  

(6)

Equation 6 motivates the block diagram of Figure 4 for representation of the contact force in the system where $V_1$ and $V_2$ are given by equations (7) and (8).

\[ V_1 = (S_1 + S_2)^{-1}G_1 \]  

(7)

\[ V_2 = (S_1 + S_2)^{-1}G_2 \]  

(8)

\[ f_2 = V_1e_1 - V_2e_2 \]  

(9)

We assume Figure 4 is valid for representation of the non-linear case also. In other words, considering equations (4) and (5) as original equations for dynamic behavior of the robots, one can arrive at operators $V_1$ and $V_2$ to show the contributions of $e_1$ and $e_2$ on the contact force. We assume $V_1$ and $V_2$ are two $L_p$-stable operators, in other words $V_1(e_1) : L_p^n \rightarrow L_p^n$ and $V_2(e_2) : L_p^n \rightarrow L_p^n$ and also there exist positive scalars $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ such that:

\[ \|V_1(e_1)\|_p \leq \alpha_1 \|e_1\|_p + \beta_1 \]  

(10)

\[ \|V_2(e_2)\|_p \leq \alpha_2 \|e_2\|_p + \beta_2 \]  

(11)

See Appendix A for some definitions on the $L_p$ stability.

**THE CLOSED-LOOP SYSTEM FOR TWO ROBOTS**

The control architecture in Figure 5 shows how we develop compliancy in the system. $H_2$ is a compensator to be designed for the second robot. The input to this compensator is the contact force, $f_2$. The compensator output signal is being added vectorially with the input command vector, $r_2$, resulting in the error signal, $e_2$ for the second robot manipulator. One can think of this architecture as a system that allows the second robot to “control” the force and the first robot to “control” the position.

There are two feedback loops in the system; the first loop (which is the natural feedback loop), is the same as the one shown in Figure 3. This loop shows how the contact force affects the robots in a natural way when two robots are in contact with each other. The second feedback loop is the controlled feedback loop.

If two robots are not in contact, then the dynamic behavior of each robot reduces to the one represented by equation (1) (with $f = 0$), which is a simple tracking system. When the robots are in contact with each other, then the contact forces and the end-point positions of robots are given by $f_1$, $f_2$, $y_1$ and $y_2$ where the following equation are true:

\[ y_1 = G_1(e_1) + S_1(f_1) \]  

(12)

\[ f_2 = S_2^{-1}(y_2 - G_2(e_2)) \]  

(13)

\[ y_1 = y_2 \]  

(14)

\[ f_1 + f_2 = 0 \]  

(15)

\[ e_2 = r_2 + H_2(f_2) \]  

(16)
If all the operators are considered linear transfer functions, then:

\[ f_2 = (S_1 + S_2 + G_2 H_2)^{-1}(G_1 e_1 - G_2 r_2) \quad (17) \]

We plan to choose a class of compensators, \( H_2 \), to control the contact force with the input command \( r_2 \). This controller must also guarantee the stability of the closed-loop system shown in Figure 5. Note that the robot sensitivity functions and the electronic compliancy, \( G_2 H_2 \), add together to form the total sensitivity of the system. If \( H_2 = 0 \), then only the sensitivity functions of two robots add together to form the compliancy of the system. By closing the loop via \( H_2 \), one can not only add to the total sensitivity but also shape the sensitivity of the system.

When two robots are not in contact with each other, the actual end-point position of each robot is almost equal to its input trajectory command governed by equation (1) (with \( f = 0 \)). When the robots are in contact with each other, the contact forces on the second robot follow \( r_2 \) according to equations (12)–(16). The input command vector, \( r_2 \), is used differently for the two categories of maneuvering of the second robot; as an input trajectory command in unconstrained space (equation (1) with \( f = 0 \)) and as a command to control the force in constrained space.

STABILITY

The objective of this section is to arrive at a sufficient condition for stability of the system shown in Figure 5. This sufficient condition leads to the introduction of a class of compensators, \( H_2 \), that can be used to develop compliancy for the class of robot manipulators that have positioning controllers. The following theorem (Small Gain Theorem)\(^6,7\) states the stability condition of the closed-loop system shown in Figure 6. A corollary is given to represent the size of \( H_2 \) to guarantee the stability of the system.

If condition I, II and III hold:

I. \( V_1 \) and \( V_2 \) are \( L_p \)-stable operators, that is \( V_1(e_1): L_p^\rightarrow L_p \) and \( V_2(e_2): L_p^\rightarrow L_p^\) and:
   
   a) \( \|V_1(e_1)\|_p \leq \alpha_1 \|e_1\|_p + \beta_1 \)
   
   b) \( \|V_2(e_2)\|_p \leq \alpha_2 \|e_2\|_p + \beta_2 \)

II. \( H_2 \) is chosen such that mapping \( H_2(f_2) \) is \( L_p \)-stable that is

   a) \( H_2(f_2): L_p^\rightarrow L_p^ \) \quad (20)
   
   b) \( \|H_2(f_2)\|_p \leq \alpha_3 \|f_2\|_p + \beta_3 \) \quad (21)

III. and \( \alpha_2 \alpha_3 < 1 \)

then the closed-loop system in Figure 6 is stable. The proof is given in Appendix A. The following corollary develops a stability bound if \( H_2 \) is selected as a linear transfer function matrix.

Corollary

The key parameter in the proposition is the size of \( \alpha_2 \alpha_3 \). According to the proposition, to guarantee the stability of the system, \( H_2 \) must be chosen such that \( \alpha_2 \alpha_3 < 1 \). If \( H_2 \) is chosen as a linear operator (the impulse response) while all the other operators are still nonlinear, then:

\[ \|H_2(f_2)\|_p \leq \gamma \|f_2\|_p \quad (23) \]

where:

\[ \gamma = \sigma_{\text{max}}(N) \]
\( \sigma_{\text{max}} \) indicates the maximum singular value, and \( N \) is a matrix whose \( ij \)th entry is \( \|H_2(\cdot)\|_1 \). In other words, each member of \( N \) is the \( L_1 \) norm of each corresponding member of \( H_2(\cdot) \) (pulse response). Considering inequality 23, the third stability condition, inequality 22, can be rewritten as:

\[ \gamma a_2 < 1 \]  
(25)

To guarantee the closed loop stability, \( \gamma a_2 \) must be smaller than unity, or, equivalently:

\[ \gamma < \frac{1}{a_2} \]  
(26)

To guarantee the stability of the closed loop system, \( H_2 \) must be chosen such that its “size” is smaller than the reciprocal of the “gain” of the forward loop mapping in Figure 6. Note that \( \gamma \) represents a “size” of \( H_2 \) in the singular value sense.

When all the operators are linear transfer function matrices one can use Multivariable Nyquist Criterion to arrive at the sufficient condition for stability of the closed loop system. This sufficient condition leads to the introduction of a class of transfer function matrices, \( H_2 \), that stabilize the family of linearly treated robot manipulators. The detailed derivation for the stability condition is given in Appendix B. Appendix C shows that the stability condition given by Nyquist Criterion is a subset of the criteria given by the Small Gain Theorem. Using the results in Appendix B, the sufficient condition for stability is given by inequality 27.

\[ \sigma_{\text{max}}(H_2) < \frac{1}{\sigma_{\text{max}}((S_1 + S_2)^{-1}G_2)} \quad \forall \omega \in (0, \infty) \]  
(27)

Similar to the nonlinear case, \( H_2 \) must be chosen such that its “size” is smaller than the reciprocal of the “size” of the forward loop mapping in Figure 7 to guarantee the stability of the closed loop system. Note that in inequality 27 \( \sigma_{\text{max}} \) represents a “size” of \( H_2 \) in the singular value sense.

* The maximum singular value of a matrix \( A \), \( \sigma_{\text{max}}(A) \) is defined as:

\[ \sigma_{\text{max}}(A) = \max_{z \neq 0} \frac{|A z|}{|z|} \]

where \( z \) is a non-zero vector and \(|\cdot|\) denotes the Euclidean norm.

Consider \( n = 1 \) (one degree of freedom system) for more understanding about the stability criterion. The stability criterion when \( n = 1 \) is given by inequality 28.

\[ |G_2 H_2| < |S_1 + S_2| \quad \forall \omega \in (0, \infty) \]  
(28)

where \(|\cdot|\) denotes the magnitude of a transfer function. Since in many cases \( G_2 \approx 1 \) with the bandwidth of the tracking controller of each robot, \( \omega_0 \), then \( H_2 \) must be chosen such that:

\[ |H_2| < |S_1 + S_2| \quad \forall \omega \in (0, \omega_0) \]  
(29)

Inequality 29 reveals some facts about the size of \( H_2 \). The smaller the sensitivity functions of the robot manipulators are, the smaller \( H_2 \) must be chosen. In the “ideal case”, no \( H_2 \) can be found to allow two perfect tracking robots \( (S_1 = S_2 = 0) \) interact with each other. In other words, for the stability of the system shown in Figure 5, there must be some compliancy in either first or second robot. RRC, structural dynamics, and the tracking controller stiffness form the compliancy on the robot.

Suppose, the first robot is an ideal positioning system. In other words, \( S_1 \) has a zero gain. Therefore the contact force and the position of contact point between two robots are:

\[ f_{2w} = (S_2 + G_2 H_2)^{-1}(G_1 e_1 - G_2 r_2) \]  
(30)

\[ y_{2w} = G_1(e_1) \]  
(31)

The first robot controls the position of the contact point, while the other controls the contact force. Generalizing this concept to \( n \) robots, one robot controls the position of the contact point while the other robots control the \( n - 1 \) contact forces such that:

\[ f_1 + f_2 + f_3 + \cdots + f_n = 0 \]  
(32)

EXAMPLE

Consider two one-degree of freedom robots with \( G \) and \( S \) in equation (1) given as:

\[ G_1(s) = \frac{0.85}{(s/5+1)(s/9+1)(s/190+1)(s/240+1)(s/290+1)} \]

\[ G_2(s) = \frac{1}{(s/6+1)(s/10+1)(s/200+1)(s/250+1)(s/300+1)} \]

\[ S_2(s) = \frac{0.05}{(s/4+1)(s/8+1)} \]

\[ S_1(s) = \frac{0.05}{(s/5+1)(s/9+1)} \]
Definition 1: For all $p \in (1, \infty)$, we label as $L_p^n$ the set consisting of all functions $f = (f_1, f_2, \ldots, f_n)^T: (0, \infty) \rightarrow \mathbb{R}^n$ such that:

$$
\int_0^\infty |f_i|^p \, dt < \infty \quad \text{for } i = 1, 2, \ldots, n
$$

Definition 2: For all $T \in (0, \infty)$, the function $f_T$ defined by:

$$
f_T = \begin{cases} 
0 & 0 \leq t \leq T \\
T & T < t
\end{cases}
$$

is called the truncation of $f$ to the interval $(0, T)$.

Definition 3: The set of all functions $f \in L^n_+: \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ such that $f \in L_p^n$ for all finite $T$ is denoted by $L^n_+$. $f$ by itself or may not belong to $L^n_+$.

Definition 4: The norm on $L_p^n$ is defined by:

$$
\|f\|_p = \left( \sum_{i=1}^n |f_i|^p \right)^{1/p}
$$

where $w_i$ is the weighting factor. $w_i$ is particularly useful for scaling forces and torques of different units.

Definition 5: Let $V_2(\cdot): L_p^n \rightarrow L_p^n$. We say that the operator $V_2(\cdot)$ is $L_p^n$-stable, if:

a) $V_2(\cdot): L_p^n \rightarrow L_p^n$

b) there exist finite real constants $\alpha_2$ and $\beta_2$ such that:

$$
\|V_2(e_2)\|_p \leq \alpha_2 \|e_2\|_p + \beta_2 \quad \forall e_2 \in L_p^n
$$

According to this definition we first assume that the operator maps $L_p^n$ to $L_p^n$. It is clear that if one does not show that $V_2(\cdot): L_p^n \rightarrow L_p^n$, the satisfaction of condition (a) is impossible since $L_p^n$ contains $L_\infty^n$. Once the mapping of $V_2(\cdot)$ from $L_p^n$ to $L_p^n$ is established, then we say that the operator $V_2(\cdot)$ is $L_p^n$-stable if whenever the input belongs to $L_p^n$ the resulting output belongs to $L_p^n$. Moreover, the norm of the output is not larger than $\alpha_2$ times the norm of the input plus the offset constant $\beta_2$.

Definition 6: The smallest $\alpha_2$ such that there exists a constant $\beta_2$ so that inequality b of Definition 5 is satisfied is called the gain of the operator $V_2(\cdot)$.

Definition 7: Let $V_2(\cdot): L_p^n \rightarrow L_p^n$. The operator $V_2(\cdot)$ is said to be causal if:

$$
V_2(e_2) = V_2(e_2T) \forall T < \infty \quad \text{and } \forall e_2 \in L_p^n
$$
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PROOF OF THE NONLINEAR STABILITY PROPOSITION
Define the closed-loop mapping \( A: (e_1, r_2) \rightarrow e_2 \) (Figure 6).

\[
e_2 = r_2 + H_2(V_1(e_1) - V_2(e_2)) \quad (A1)
\]

For each finite \( T \), inequality A2 is true.

\[
\|e_{2T}\|_p \leq \|r_{2T}\|_p + H_2(V_1(e_1) - V_2(e_2))_T \|_p \quad \forall T < \infty
\]

Therefore, using inequalities 18, 19, and 21:

\[
\|e_{2T}\|_p \leq \|r_{1T}\|_p + \alpha_3 \alpha_1 \|e_{1T}\|_p + \alpha_3 \alpha_2 \|e_{2T}\|_p + \alpha_3 \beta_1 + \alpha_3 \beta_2 + \beta_3 \quad \forall T < \infty \quad (A3)
\]

Since \( \alpha_3 \alpha_2 \) is less than unity:

\[
\|e_{2T}\|_p \leq \|r_{2T}\|_p + \frac{\alpha_3 \beta_1}{1 - \alpha_3 \alpha_2} + \frac{\alpha_3 \beta_2 + \beta_3}{1 - \alpha_3 \alpha_2} \quad \forall T < \infty \quad (A4)
\]

Inequality A4 shows that \( e_2(t) \) is bounded over \( (0, T) \). Because this reasoning is valid for every finite \( T \), it follows that \( e_2(t) \in L_{pe} \), i.e., that \( A: L_{pe} \rightarrow L_{pe} \). Next we show that the mapping \( A \) is \( L_{pe} \)-stable in the sense of Definition 5. Since \( \|r_2\|_p \) and \( \|e_1\|_p < \infty \) (they both belong to \( L_{pe} \) space), then from inequality A4:

\[
\|e_{2T}\|_p < \infty \quad \forall T < \infty \quad (A5)
\]

In the limit when \( T \rightarrow \infty \):

\[
\|e_2\|_p < \infty \quad (A6)
\]

Inequality A6 implies \( e_2 \) belongs to \( L_{pe} \)-space whenever \( r_2 \) and \( e_1 \) belong to \( L_{pe} \)-space. With the same reasoning from equation (A1) to (A5), it can be shown that inequality (A7) is true.

\[
\|e_2\|_p \leq \|e_1\|_p + \frac{\|r_{2T}\|_p}{1 - \alpha_3 \alpha_2} + \frac{\alpha_3 \beta_1 + \beta_3}{1 - \alpha_3 \alpha_2} \quad (A7)
\]

Inequality (A7) shows the linear boundedness of \( e_2 \). (Condition b of Definition 5). Inequality (A7) and (A6) taken together, guarantee that the closed-loop mapping \( A \) is \( L_{pe} \)-stable.

APPENDIX B
The objective is to find a sufficient condition for stability of the closed-loop system in Figure 7 by Nyquist Criterion. The block diagram in Figure 7 can be reduced to the block diagram in Figure B1 when all the operators are linear transfer function matrices.

There are two elements in the feedback loop; \( G_2H_2S_1^{-1} \) and \( S_2S_1^{-1} \cdot S_2S_1^{-1} \) shows the natural force feedback while \( G_2H_2S_1^{-1} \) represents the controlled force feedback in the system. The objective is to use Nyquist Criterion\(^8\) to arrive at the sufficient condition for stability of the system when \( H_2 = 0 \). The following conditions are noted:

1) The closed loop system in Figure B1 is stable if \( H_2 = 0 \). This condition simply states the stability of two robot manipulators. (Figure 4 shows this configuration.)

2) \( H_2 \) is chosen as a stable linear transfer function matrix. Therefore the augmented loop transfer function \( (G_2H_2S_1^{-1} + S_2S_1^{-1}) \) has the same number of unstable poles that \( S_2S_1^{-1} \) has. Note that in many cases \( S_2S_1^{-1} \) is a stable system.

3) Number of poles on \( j\omega \) axis for both loops \( S_2S_1^{-1} \) and \( (G_2H_2S_1^{-1} + S_2S_1^{-1}) \) are equal.

Considering that the system in Figure B1 is stable when \( H_2 = 0 \) we plan to find how robust the system is when \( G_2H_2S_2^{-1} \) is added to the feedback loop. If the loop transfer function \( S_2S_1^{-1} \) (without compensator, \( H_2 \)) develops a stable closed-loop system, then we are looking for a condition on \( H_2 \) such that the augmented loop transfer function \( (G_2H_2S_1^{-1} + S_2S_1^{-1}) \) guarantees the stability of the closed-loop system. According to the Nyquist Criterion, the system in Figure B1 remains stable if the anti-clockwise encirclement of the det \((G_2H_2S_1^{-1} + S_2S_1^{-1} + I_n)\) around the center of the \( s \)-plane is equal to the number of unstable poles of the loop transfer function \( (G_2H_2S_1^{-1} + S_2S_1^{-1}) \). According to conditions 2 and 3, the loop transfer functions \( S_2S_1^{-1} \) and \( (G_2H_2S_1^{-1} + S_2S_1^{-1}) \) both have the same number of unstable poles. The closed-loop system when \( H_2 = 0 \) is stable according to condition 1; the encirclements of \( \det(S_2S_1^{-1} + I_n) \) is equal to unstable poles of \( S_2S_1^{-1} \). Since the number of unstable poles of \( (G_2H_2S_1^{-1} + S_2S_1^{-1}) \) and that of \( S_2S_1^{-1} \) are the same, therefore for stability of the system \( \det(G_2H_2S_1^{-1} + S_2S_1^{-1} + I_n) \) must have the same number of encirclements that \( \det(S_2S_1^{-1} + I_n) \) has. A sufficient condition to guarantee the equality of the number of encirclements of \( \det(G_2H_2S_1^{-1} + S_2S_1^{-1} + I_n) \) and that of \( \det(S_2S_1^{-1} + I_n) \) is that \( \det(G_2H_2S_1^{-1} + S_2S_1^{-1} + I_n) \) does not pass through the origin of the \( s \)-plane for all possible non-zero but finite values of \( H_2 \), or

\[
\det(G_2H_2S_1^{-1} + S_2S_1^{-1} + I_n) \neq 0 \quad \forall \omega \in (0, \infty) \quad (B1)
\]
If inequality B1 does not hold then there must be a non-zero vector \( z \) such that:

\[
(G_2 H_2 S_{11}^{-1} + S_2 S_{11}^{-1} + I_n) z = 0 \quad \text{(B2)}
\]

or

\[
G_2 H_2 S_{11}^{-1} z = -(S_2 S_{11}^{-1} + I_n) z \quad \text{(B3)}
\]

A sufficient condition to guarantee that equality B3 will not happen is given by inequality B4.

\[
\sigma_{\max}(G_2 H_2 S_{11}^{-1}) < \sigma_{\min}(S_2 S_{11}^{-1} + I_n) \quad \forall \omega \in (0, \infty) \quad \text{(B4)}
\]

or a more conservative condition:

\[
\sigma_{\max}(H_2) < \frac{1}{\sigma_{\max}((S_1 + S_2)^{-1} G_2)} \quad \forall \omega \in (0, \infty) \quad \text{(B5)}
\]

Note that \((S_1 + S_2)^{-1} G_2\) is the transfer function matrix that maps \( e_1 \) to the contact force, \( f_2 \) when \( e_1 = 0 \). Figure 7 shows the closed-loop system. According to the result of the proposition, \( H_2 \) must be chosen such that the size of \( H_2 \) is smaller than the reciprocal of the size of the forward loop transfer function, \((S_1 + S_2)^{-1} G_2\).

**APPENDIX C**

The following inequalities are true when \( p = 2 \) and \( H_2 \) and \( V_2 \) are linear operators.

\[
\|H_2(f_2)\|_p \leq \|f_2\|_p \quad \text{(C1)}
\]

\[
\|V_2(e_2)\|_p \leq \mu \|e_2\|_p \quad \text{(C2)}
\]

where:

\[
\mu = \sigma_{\max}(Q), \quad \text{and} \quad Q \text{ is the matrix whose } ij \text{th entry is given by } (Q)_{ij} = \sup_{\omega} \|V_2)_{ij}^{\omega} |
\]

\[
\nu = \sigma_{\max}(R), \quad \text{and} \quad R \text{ is the matrix whose } ij \text{th entry is given by } (R)_{ij} = \sup_{\omega} \|H_2)_{ij}^{\omega} |
\]

According to the stability condition, to guarantee the closed loop stability \( \mu \nu < 1 \) or:

\[
\nu < \frac{1}{\mu} \quad \text{(C3)}
\]

Note that the following are true;

\[
\sigma_{\max}(V_2) \leq \mu \quad \forall \omega \in (0, \infty) \quad \text{(C4)}
\]

\[
\sigma_{\max}(H_2) \leq \nu \quad \forall \omega \in (0, \infty) \quad \text{(C5)}
\]

Substituting C4 and C5 inequality C3 which guarantees the stability of the system, the following inequality is obtained:

\[
\sigma_{\max}(H_2) < \frac{1}{\sigma_{\max}(V_2)} \quad \forall \omega \in (0, \infty)
\]

\[
\sigma_{\max}(H_2) < \frac{1}{\sigma_{\max}((S_1 + S_2)^{-1} G_2)} \quad \forall \omega \in (0, \infty) \quad \text{(C7)}
\]

Inequality C7 is identical to inequality 27. This shows that the linear stability condition by the multivariable Nyquist Criterion is a subset of the general stability condition given by the Small Gain Theorem.

**References**


