Theory and Experiments on Tracking of the Repetitive Signals

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Abstract

The work presented here is a simple feedback controller methodology that allows for exact tracking of the sinusoidal input signals and rejection of the sinusoidal disturbances in a closed loop control system. The control method is motivated by a mathematical inequality that expresses the tracking and disturbance rejection requirements for a closed loop system. The exact tracking of the input command at a particular frequency requires an infinite loop gain for the system at the frequency of the input command. A second order undamped transfer function is cascaded to teach input channel to increase the loop transfer function gain at the frequency of the input command. A feedback controller is then designed via the LQG/LTR method to stabilize the system while the loop gain remains large at the frequency of the input. The method is experimentally verified on a single axis servo system and extended to multivariable systems.

1. Introduction

Tool positioning to generate non-circular cross sections using a lathe is the reason for the work presented here. To generate a non-circular cross section in a part, one needs to move the cutting tool back and forth with high speed. By controlling the tool position in the direction normal to the surface of workpiece, one can remove appropriate amounts of metal from the part allowing for the production of parts with non-circular cross sections [7]. Tomizuka et. al. have used a zero phase error tracking control algorithm to design controllers that allow for a zero phase angle in command following [8]. A feedback compensator has been utilized for stabilization and disturbance rejection while a model-based feedforward compensator has been used for phase compensation. The system's performance strongly depends on the integrity of the feedforward compensator and consequently on the precision of the modeling. For fast tracking capability (in particular at higher frequencies), this method requires a good model of the system at higher frequencies. In another approach, repetitive controller is employed where the object is to achieve asymptotic tracking of periodic signals. In this method, the controller contains the structure of the repetitive signals cascaded to the compensated plant [10]. This approach has resulted in a narrow bandwidth (yet very robust to model parameter mismatch) due to the internal feedback of the compensated plant. A repetitive controller design method in the digital domain is given by Tomizuka et. al. for SISO systems where the controller contains a stable structure of the inverse of the plant in addition to the structure of the repetitive signals [9].

The control method described here uses a feedback compensator for exact tracking of the sinusoidal input commands with a known frequency. The controller parameters depend on the frequency of the input command. The feedback controller is designed so the loop transfer function gain becomes large for the desired frequency of input. This controller allows not only for tracking of the signals, but also for rejection of the disturbances at a particular frequency. One can think of this method as a generalization of "notch filter" used in practice for disturbance rejection and noise reduction. The controller contains the structure of the command signal in addition to the local inverse of the plant which has been achieved by LQG/LTR formalism [1, 3, 5]. We explicitly show the limitations that might arise from unmodeled dynamics, noise and right half plane zeros. We also show that the technique is robust in the presence of parameter uncertainties in modeling the plant.

We start with the SISO systems tracking requirements where a frequency dependent lower bound has been derived for the loop transfer function gain. This bound narrows the choice of tracking compensator via the LQG/LTR method. We then experimentally demonstrate the effectiveness of the control method and lastly we extend the theoretical results to multivariable systems that must follow exactly a vector of sinusoidal inputs with one single frequency component.

2. Control Architecture

We start with a standard SISO feedback configuration as shown in Figure 1. It consists of interconnected plant G(s), and controller K(s) forced by command R(s), and disturbances D(s). All disturbances are assumed to be reflected at the output of the plant. Note that all disturbances that arrive in the loop at the input to the plant can always be reflected at the output of the plant by proper dynamic scaling of the disturbance. The optional pre-compensator, P(s), is used to calibrate the input command. Both models for the plant and the controller are rational transfer functions.

![Figure 1: The Standard Closed Loop System](image)
We propose to design the compensator $K(s)$, such that the closed loop system shown in Figure 1 is stable and allows for tracking of the input commands that are bounded in magnitude and frequency very "closely" for all the frequency range of the input. We define the closeness of the system output to input command via satisfaction of the following inequality when $N(s)D(s)=0$:

$$\left|\frac{V(j\omega)}{R(j\omega)} - \frac{G_u(j\omega)}{G(j\omega)}\right| < \varepsilon_R \text{ for all } \omega \in [0, \omega_R]$$  \hspace{1cm} (1)

where $\omega_R$ is the frequency range of input. Note that $\varepsilon_R$, (tracking error) expressed by the designer, is a small number that shows the closeness of the output to input. e.g. for a system with good tracking capability, $\varepsilon_R$ could be as low as 0.05 and that indicates the output follows the input with 0.05 error). Inequality 1 does not imply any controller design method; it only allows the designers to express what they want to have in a closed loop system for tracking signals for all $\omega \in [0, \omega_R]$. It is important to note that $N(s)$ is a finite bandwidth of the closed loop system. The wider $N(s)$ is chosen to be, the faster the closed loop system is expected to be. This bandwidth is limited by the amount of unmodeled dynamics in the plant [6]. Referring to Figure 1, the error signal $E(s)$, is:

$$E(s) = \frac{1}{G(s)K(s)} (R(s) - \frac{1}{1+G(s)K(s)} D(s)$$

$$+ \frac{G(s)K(s)}{1+G(s)K(s)} N(s))$$  \hspace{1cm} (2)

Considering equation 2, when $N(s)D(s)=0$, inequality 1 can be written as:

$$\left|\frac{1}{1+G(j\omega)K(j\omega)}\right| < \varepsilon_R \text{ for all } \omega \in [0, \omega_R]$$  \hspace{1cm} (3)

If the compensator, $K(s)$, is designed such that inequality 3 is satisfied, the output follows all the frequency components of the input command within the precision of $\varepsilon_R$. A more conservative inequality can be given on the loop transfer function, $G(j\omega)K(j\omega)$, by manipulation of inequality 3:

$$|G(j\omega)K(j\omega)| > 1$$

Since for good tracking systems, $\varepsilon_R$ is a small number, the loop transfer function gain, $|G(j\omega)K(j\omega)|$ must be very large for all $\omega \in [0, \omega_R]$ to guarantee the closeness of the output to input. The smaller $\varepsilon_R$ is selected to be, the larger the loop transfer function must be for all $\omega \in [0, \omega_R]$. In the limit, when $\varepsilon_R=0$, one requires an infinitely large loop transfer function for the system. Hereafter, we focus on the tracking signals at a particular and known frequency of interest namely, $\omega_o$, rather than the whole frequency range $[0, \omega_R]$. In other words, we are interested to arrive at a controller that is able to track a sinusoidal signal very closely.

Suppose one wishes to track a sinusoidal input with frequency of $\omega_o$, with zero error. We cascade the plant by a second order under damped transfer function, $G_o(s)$, to increase the magnitude of the loop transfer function at $\omega_o$ as required by inequality 4 (Figure 2).

$$G_o(s) = \frac{\omega_o^2}{s^2 + \omega_o^2}$$  \hspace{1cm} (5)

Since $G_u(s)$ has a zero damping, the magnitude of the augmented loop transfer function $G_u(s)G(s)$, is very large at $\omega_o$. However this renders a highly unstable open loop system. We propose to consider the augmented plant dynamics as a new plant dynamics to be stabilized by feedback. We use the LQG/LTR to design $K(s)$ to stabilize the closed loop system of Figure 2. Note that the nature of $K(s)$ is not of significance at this stage, however $K(s)$ will be an "ideal" inverse of the plant, when LQG/LTR has been used for the design procedure.

The closed loop transfer function $V(s)$ can be obtained from the LQG/LTR as follows:

$$V(s) = G(s)K(s)G_o(s)$$

$$R(s) = \frac{G_o(s)K(s)}{1+G_o(s)K(s)} D(s)$$  \hspace{1cm} (7)

Equation 7 results in a unity magnitude at $\omega=\omega_o$. In other words, the plant output follows the input exactly if the input is a sinusoidal function with a frequency of $\omega_o$. Note that the exact tracking at other frequencies is not guaranteed. The relationship between the disturbance and the output, $V(s)$, is given by the sensitivity transfer function of equation 8.

$$\frac{V(s)}{D(s)G_o(s)K(s)} = \frac{G_o(s)K(s)}{G(s)K(s)} + G_o(s)K(s)$$

At $\omega=\omega_o$, the magnitude of the sensitivity transfer function will be zero, thus giving rise to perfect disturbance rejection at $\omega=\omega_o$.

To summarize, for tracking sinusoidal signals very "closely" (or rejecting the sinusoidal disturbances) at a particular frequency, $\omega_o$, the loop gain at that frequency should be very large. The augmentation of the loop transfer function by an undamped second order transfer function leads to (theoretically) perfect tracking and disturbance rejection at the frequency equal to the natural frequency of the undamped second order transfer function. In the next section, an experiment is described to show the strength of the method in tracking a sinusoidal input. We then extend the theoretical results to the multivariable systems in Sections 4 and 5.

3. Experimental Results

The system consists of a link driven by a DC servo motor shown in Figure 3.
Figure 3: The Plant Configuration

where:

\[ R : \text{terminal resistance (3.09 ohms)} \]
\[ K_1 : \text{gain at the positive summing junction of servo controller (1.56 v/v)} \]
\[ K_2 : \text{amplifier gain at the negative summing junction of servo (56.4 v/v)} \]
\[ K_T : \text{motor torque constant (0.63 Newton Meter / amp)} \]
\[ K_E : \text{back emf constant (0.063 volts / rad / sec)} \]
\[ J : \text{moment of inertia (.0014 kg - meter}^2) \]
\[ V_y : \text{velocity feed back gain (0.0002 volts / rad / sec)} \]
\[ U : \text{input command to motor (volts)} \]
\[ Y : \text{orientation of shaft in radians} \]
\[ I : \text{current in amps} \]

\[
G(s) = \frac{V(s)}{U(s)} = \frac{K_1 K_2 K_T}{R J} \frac{1}{s(s + K_T K_F + K_2 K_T K_V)}
\]

Inserting the values for all the system parameters, the system transfer function from input command, \( U \), to the output orientation, \( Y \), will be equal to:

\[
G(s) = \frac{12813}{s \left( \frac{s}{408.5 \cdot 10^3} + 1 \right)}
\]

The plant dynamics is augmented by the addition of a second order filter with zero damping. The natural frequency of the filter is chosen to be the frequency of the sinusoidal input command (or the frequency of the disturbance signals) acting on the system. The new plant transfer function is:

\[
G_0(s) G(s) = \frac{\omega_0^2}{s^2 + \omega_0^2} \frac{12813}{s \left( \frac{s}{408.5 \cdot 10^3} + 1 \right)}
\]

where \( \omega_0 \) is chosen to be 200 rad/sec (30 Hertz). Note that the order of the new plant has increased by the augmentation of the extra dynamics to the plant. The compensator \( K(s) \) (equation 12) was designed via the LQG/LTR method to guarantee the closed loop stability of the system and robustness in the presence of bounded unmodeled dynamics. (This will be explained in Sections 4 and 5).

\[
K(s) = \frac{0.6653}{\left( \frac{s}{135.4 \cdot 10^3} + 1 \right) \left( \frac{s}{53865} + 1 \right)}
\]

The closed loop transfer function of the system will be:

\[
G_{\text{closed}}(s) = \frac{1}{\left( \frac{s}{34} + 1 \right) \left( \frac{s}{93} + 1 \right) \left( \frac{s}{498} + 1 \right)} \left( \frac{s^2}{\left( \frac{s}{410.30} + 1 \right) \left( \frac{s}{1340} + 1 \right) \left( \frac{s}{47513} + 1 \right) \left( \frac{s}{576} + 1 \right)} \right)
\]

The bode plot in Figure 4 and 5 show the experimental and theoretical closed loop transfer function gain and phase with unity magnitude and -360 degree phase at \( \omega_0 = 200 \text{ rad/sec} \). Figure 6 shows the loop transfer function where its value is large at \( \omega_0 = 2000 \text{ rad/sec} \). The loop transfer function crosses over the zero db line before the frequency range of the unmodeled dynamics. Figure 7 shows the theoretical value of the sensitivity transfer function where a "notch" filter type compensator is developed in the system for tracking.

Figure 8 shows the experimental time domain frequency response for various values of the input frequencies. Figure 8a shows the steady state response of the system when the input command is a 1.6 hertz (about 10 rad/sec) sinusoidal function. At low frequencies the system follows the input with little delay. Figure 8b shows the system response when input frequency is 16 hertz (about 100 rad/sec). The system output has been attenuated and has a considerable amount of delay. Figure 8c shows the case when the input frequency is about 30 hertz (200 rad/sec). The amplitude of the system output is almost equal to the amplitude of the input and the delay is about 360. Figure 8d shows a case where the input command is 35 hertz (220 rad/sec) and the system output has been attenuated. These time domain experiments show that the system follows the input command with less precision at frequencies other than 30 hertz.
4. Extension to Multivariable Systems

The design methodology can be extended to multivariable systems. We insert a second order underdamped transfer function for each input channel to increase the loop gain at $\omega = \omega_0$ (Figure 9). This results in a diagonal augmenting transfer function matrix of the following form:

$$G_p(s) = \frac{\omega_0^2}{s^2 + \omega_0^2}$$  \hspace{1cm} (14)

One must guarantee the following inequality to track the sinusoidal signals with precision of $\varepsilon_R$.

$$\sigma_{max} \left[ 1 + G(j \omega) \cdot G_p(j \omega) \cdot K(j \omega) \right]^{-1} \leq \frac{\varepsilon_R}{\varepsilon_R} \text{ for } \omega = \omega_0$$  \hspace{1cm} (15)

where $\sigma_{max}$ indicates the maximum singular value of the transfer function and $K(s)$ is the stabilizing compensator.

A more conservative condition for tracking with the

$3$The maximum singular value of a matrix $A$, $\sigma_{max}(A)$ is defined as:

$$\sigma_{max}(A) = \max \left| \frac{A \cdot z}{z} \right|$$

where $z$ is a non-zero vector and $\|z\|$ denotes the Euclidean norm.

The precision of $\varepsilon_R$ is given by inequality 16.

$$\sigma_{max} \left[ G(j \omega) \cdot G_p(j \omega) \cdot K(j \omega) \right]^{-1} \leq \frac{1}{\varepsilon_R} \text{ for } \omega = \omega_0$$  \hspace{1cm} (16)

Figure 8: The Experiment on Command Following, $\omega_0 = 30$ hertz.

-, - - - - : input command, --- : measured output position

By inspection of Figure 9, the closed loop transfer function matrix is:

$$G_{closed}(s) = G_p(s)K(s) \left[ I_n + G_p(s)K(s) \right]^{-1}$$  \hspace{1cm} (17)

where $G_p(j \omega) = G(j \omega)G_p(j \omega)$. It can be shown that size of the loop transfer function, $G_p(s)K(s)$, approaches infinity in the singular value sense and $G_{closed}(s)$ approaches the unity matrix when $\omega = \omega_0$. This leads to the exact command following of the input commands at $\omega_0$. Limitations in the design of the compensator, $K(s)$, may arise when the desired tracking frequency, $\omega_0$, lies within or in the proximity of the frequency range of the unmodeled dynamics.

One always has errors due to intentional approximation of higher order dynamics by lower order
5. Design Method

The objective is to design a compensator, \( K(s) \) such that \( G_p(j\omega)K(j\omega) \) will satisfy inequality 18. The augmented plant \( G_p(j\omega) \) is first realized in the state space form with four matrices of \( A, B, C \) and \( D \). One traditional method of designing \( K(s) \) consists of two stages. The first stage concerns state feedback gain design. A state feedback gain \( F \) is designed so that the state feedback closed loop system is stable or equivalently:

\[
\begin{align*}
\lambda_1 & \leq 0, & u_1 & = 0, & 1 & = 1, 2, \ldots, n \quad (19) \\
\text{real}(\lambda_i) & < 0, & u_i & = 0, & i & = 1, 2, \ldots, n.
\end{align*}
\]

where \( \lambda_i \) and \( u_i \) (\( i = 1, 2, \ldots, n \)) are closed loop eigenvalues and eigenvectors of state feedback configuration.

In the second stage, an observer gain \( H \) is designed to make the first stage realizable. For stability of the observer the following inequality must be guaranteed:

\[
\nu_i^T (\mu_i I_n - A + HC) \gamma_n \geq 0, \quad i = 1, 2, \ldots, n. \quad (20)
\]

where \( \mu_i \) and \( \nu_i^T \) (\( i = 1, 2, \ldots, n \)) are the closed loop eigenvalues and left eigenvectors of the observer. Combining the state feedback and observer designs (Figure 10) yields the unique compensator transfer function matrix given by

\[
K(s) = G(sI - A + BF + HC)^{-1}H
\]

In this paper we take a different approach in design of \( H \) and \( F \). First we design a stable \( H \) such that the loop transfer function \( C(sI - A)^{-1}H \) meets inequality 16. In the second stage of the compensator design a stable state feedback gain \( F \) is designed to guarantee that the final loop transfer function \( G_p(s)K(s) \) maintains the same loop shape that \( C(sI - A)^{-1}H \) achieved via filter design at the first stage. This is the principle behind Loop Transfer Recovery.

![Figure 10: Closed Loop System](image)

Figure 9: The Closed Loop Block Diagram for the Multivariable System

The eigenstructure properties of LTR for a general multi-input multi-output system have been discussed in reference [5, 6]. Here we summarize the properties of the Loop Transfer Recovery. If \( F \) is chosen such that limit 22 is true as \( \rho \) approaches zero for any non-singular (m\( \times \)m) \( W \) matrix,

\[
\sqrt{\rho} F \rightarrow W C
\]

then \( K(s) \) approaches pointwise (non-uniformly) towards expression 23.4

\[
[C(sI - A)^{-1} B]^{-1} C(sI - A)^{-1} H
\]

4 This result shows how LQG/LTR develops a "legal" inverse of the plant.
and since \( G_\phi(s) = C(sI - A)^{-1}B \), then \( G_\phi(s) K(s) \) will approach \( C(sI - A)^{-1}H \) non-uniformly and the followings are true:

1) The finite transmission zeros \( \{ \} \) of the compensator \( K(s) \) are the same as the finite transmission zeros of \( C(sI - A)^{-1}H \).

2) All the transmission zeros of the compensator \( K(s) \), cancel the transmission zeros (including ones at infinity) of the plant.

According to property 2, as \( \rho \) approaches zero, the eigenvalues of \( K(s) \) will cancel out the transmission zeros of the plant. According to property 1, as \( \rho \) approaches zero, the transmission zeros of \( K(s) \) will approach the transmission zeros of \( C(sI - A)^{-1}H \). Since the number of transmission zeros of two cascaded systems \( K(s) \) and \( G(s) \) is the sum of the number of transmission zeros of both systems, the transmission zeros of \( G(s)K(s) \) are the same as the transmission zeros of \( C(sI - A)^{-1}H \). Similar arguments can be given for the poles of \( G(s)K(s) \). The poles of \( K(s) \) cancel out the transmission zeros of the plant; therefore the poles of \( G(s)K(s) \) will be the same as the poles of \( C(sI - A)^{-1}H \). This argument does not prove the asymptotic equality of \( C(sI - A)^{-1}H \) and \( G(s)K(s) \) as \( \rho \) approaches zero. Proof of the pointwise equality of \( G(s)K(s) \) and \( C(sI - A)^{-1}H \) is best shown in [3].

The above comment concerning pole-zero cancellation explains the eigenstructure mechanism for LTR. Since pole placement and eigenvector construction in the allowable subspace prescribes a unique value for \( F \), one may design the \( F \) via pole placement and eigenvector construction.

Difficulty in using LTR will arise if the plant has some right, half-plane zeros (non-minimum-phase plant).

In our proposed procedure for LTR, one should place the eigenvalues of \( A-BF \) at the transmission zeros of the plant. If the plant is in non-minimum phase, one would place some eigenvalues of \( A-BF \) on the right half-plane. The closed-loop system will not be stable if any eigenvalues of \( A-BF \) are on the right half-plane. According to the separation theorem, the eigenvalues of \( A-BF \) are also the eigenvalues of the closed-loop system. Therefore the sufficient condition for LTR and the stability of the closed-loop system is that the plant be minimum-phase. If the plant is non-minimum phase, one should consider the mirror images of the right half-plane zeros as target locations for eigenvalues of \( A-BF \). In such cases, loop transfer recovery is not guaranteed, but the closed-loop system will be stable. The limitation associated with non-minimum phase transmission zeros which has also been seen in \( H^\infty \) design method was described in the best in reference [4].

The above design method has been used for the design of the compensator in equation 12. As the experimental values of the loop transfer function show in figure 6, the unmodeled dynamics in plant starts at the about 350 rad/sec. Using \( F = \{-32, .05, 1.11, .002\} \) and \( H = \{-750, 323, 203, 1763\} \), the compensator \( K(s) \) of equation 12 results.

This compensator allows for a loop transfer function that is below the zero db line within the frequency range of unmodeled dynamics. The compensator has three large poles to approximately cancel out the three infinite zeros of the plant. The two zeros of the compensator add lead angle to the loop transfer function to generate stability.

6. Conclusion

The work presented here is a simple and practical control methodology in the design of compensators that allow for tracking input signals at a particular frequency. The tracking capability at a particular frequency requires a large loop gain at that frequency. We use an undamped transfer function as a pre-compensator to guarantee a large size of the loop transfer function. The loop transfer function is then stabilized by a feedback compensator within the \( LQG/LTR \) formalism. A limitation will arise when the tracking frequency is within or very close to the unmodeled dynamics of the system.

References