

## On the loop transfer recovery

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One method of model-based compensator design for linear multivariable systems consists of state-feedback design and observer design (Athans 1971). A key step in recent work in multivariable synthesis involves selecting an observer gain so the final loop-transfer function is the same as the state-feedback loop transfer function (Doyle and Stein 1979, Goodman 1984). This is called loop transfer recovery (LTR). This paper shows how identification of the eigenstructure of the compensators that achieve LTR makes possible a design procedure for observer gain (Luenberger 1971). This procedure is based on the eigenstructure assignment of the observers. The sufficient condition for LTR and the stability of the closed-loop system is that the plant be minimum-phase. The limitation of this method might arise when the plant has multiple transmission zeros.

### Nomenclature

$A, B, C$	plant parameters
$d_i(t), d_o(t)$	input and output disturbances
$x(t), u(t), y(t)$	states, input and output of the system
$\hat{x}(t), \hat{y}(t)$	states and output of the observers
$e(t)$	error signal of the observer
$G_P(s)$	transfer-function matrix of the plant
$\lambda_i$	eigenvalues of $A - BG$
$\mu_i$	eigenvalues of $A - HC$
$s_i$	transmission zeros of $A, B, C$
$\sigma_i$	transmission zeros of $A, B, G$
$G$	state-feedback gain
$K(s)$	transfer-function matrix of the compensator
$\rho$	positive scalar
$v_i^T$	left-eigenvector of $A - HC$
$u_i$	right-eigenvector of $A - BG$
$W$	square non-singular $m \times m$ matrix
$z_i^T$	zero direction of the transmission zero
$w_i^T$	input direction of the transmission zero
$j$	maximum number of the finite transmission zeros
$x_i^T$	left-eigenvector of $A - BG - HC$
$\Phi_{oi}(s)$	open-loop characteristic equation of the plant
$\Phi_{ci}(s)$	closed-loop characteristic equation of the observer
$n$	order of the system
$m$	rank of matrices $B$ and $C$
$P(s)$	precompensator

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### 1. Introduction

Historically, the LTR method is the consequence of attempts by Doyle and Stein (1979, 1981) to improve the robustness of linear quadratic gaussian (LQG) regulators. In their seminal work, Doyle and Stein address the problem of finding the steady-state observer gain that assures the recovery of the loop transfer function resulting from full state feedback. First, they demonstrate a key lemma that gives a sufficient condition for the steady-state observer gain such that LTR takes place. To compute the gain, they show that the infinite time-horizon Kalman-filter formalism with 'small' white measurement-noise covariance yields an observer gain that satisfies the sufficient condition for loop-transfer recovery. In this paper we present a method for computing observer gain that obviates the need for Kalman-filter formalism. The goal of this paper is to analyse the eigenstructure properties of the LTR method for the general class of feedback control systems that use model-based compensators. After examining the eigenstructure of LTR, a design methodology for LTR via eigenstructure assignment will be given.

### 2. Background

We will deal with the standard feedback configuration shown in Fig. 1, which consists of: plant model  $G_p(s)$ ; compensator  $K(s)$ , forced by command  $r(t)$ ; measurement noise  $n(t)$ ; and the disturbances  $d_i(t)$  and  $d_o(t)$ . The precompensator  $P(s)$ , is used to filter the input for command-following.

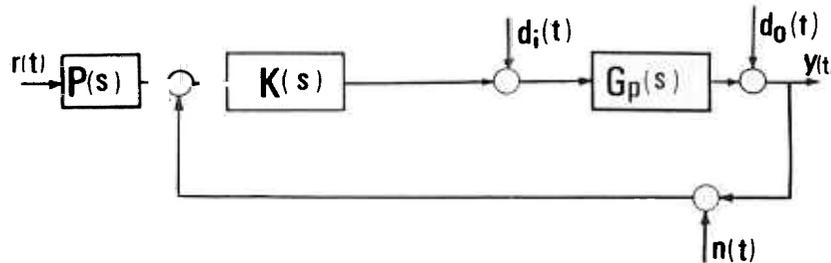


Figure Standard closed-loop system.

Throughout this paper, we assume that the plant can be described by the following equations:

$$\dot{x}(t) = Ax(t) + Bu(t) + Bd_1(t) \quad (1)$$

$$y(t) = Cx(t) + d_o(t) + n(t) \quad (2)$$

where

$$x(t) \in R^n, \quad u(t), y(t), d_1(t), d_o(t), n(t) \in R^m$$

$[A, B]$  is a stabilizable (controllable) pair

$[A, C]$  is a detectable (observable) pair

$$\text{rank}(B) = \text{rank}(C) = m$$

Once we specify the plant model  $G_p(s)$ , we must find  $K(s)$  so that:

- (i) the nominal feedback design  $y(s) = G_p(s)[I_{mm} + K(s)G_p(s)]^{-1}d_1(s)$  is stable;
- (ii) the perturbed system in the presence of bounded unstructured uncertainties is stable;
- (iii) application-dependent design specifications are achieved.

The design specifications can be expressed as frequency-dependent constraints on the loop transfer function  $K(s)G_p(s)$ . The standard practice is to shape the loop transfer function,  $K(s)G_p(s)$  so it does not violate the frequency-dependent constraints (Doyle and Stein 1981). The loop-shaping problem can be considered to be a design trade-off among performance objectives, stability in the face of unstructured uncertainties (Lehtomaki *et al.* 1981, Safonov and Athans 1977) and performance limitations imposed by the gain/phase relationship. Here we assume that  $n(t)$  is a noise signal that operates over a frequency range beyond the frequency range of  $r(t)$ ,  $d_i(t)$  and  $d_o(t)$ . We also use a precompensator  $P(s)$  to shape the input for command-following. Therefore the performance objectives are considered as only input-disturbance rejection over a bounded frequency range. The design specifications may be frequency-dependent constraints on  $G_p(s)K(s)$ , which is the loop transfer function broken at the output of the plant, rather than on  $K(s)G_p(s)$ , which is the loop transfer function broken at the input to the plant. Since Doyle and Stein first applied LTR to the loop transfer function  $K(s)G_p(s)$ , for consistency and continuity, we will also assume throughout this article that all design specifications apply to  $K(s)G_p(s)$ .

One method of designing  $K(s)$  consist of two stages. The first stage concerns state-feedback design. A state-feedback gain  $G$  is designed so that the loop transfer function  $G(sI_{nn} - A)^{-1}B$ , which is shown in Fig. 2, meets the frequency-dependent design specifications and satisfies the following to guarantee stability:

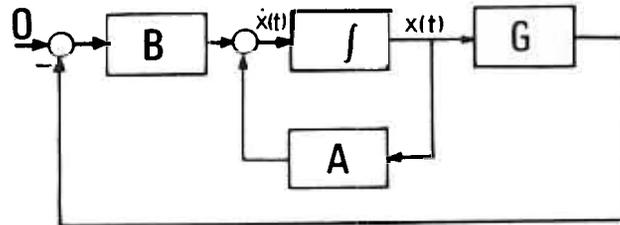


Figure 2. State-feedback configuration.

$$(\lambda_i I_{nn} - A + BG)u_i = 0_n, \quad i = 1, 2, \dots, n \quad (3)$$

$$\text{real}(\lambda_i) < 0, \quad u_i \neq 0_n$$

$\lambda_i$  is the closed-loop state-feedback eigenvalue, while  $u_i$  is the  $n \times 1$  right closed-loop eigenvector of the system. Controllability of  $[A, B]$  guarantees the existence of  $G$  in (3). At this stage, one can determine whether or not state-feedback design can meet the design specifications. In this paper we assume that  $G$  is selected so that (3) is satisfied and the loop transfer function  $G(sI_{nn} - A)^{-1}B$ , which is shown in Fig. 2, meets the desired frequency-domain design specification. In the second-stage of the compensator design, an observer is designed to make the first stage realizable (Luenberger 1966, Yuksel and Bongiorno 1971). Figure 3 shows the structure of the closed-loop observer. The observer design is not involved in meeting the specifications for the loop transfer function, since all design specifications have been met by the state feedback gain  $G$ . For stability of the observer, the following must also be satisfied:

$$v_i^T(\mu_i I_{nn} - A + HC) = 0_n^T, \quad i = 1, 2, \dots, n \quad (4)$$

$$\text{real}(\mu_i) < 0, \quad v_i^T \neq 0_n^T$$

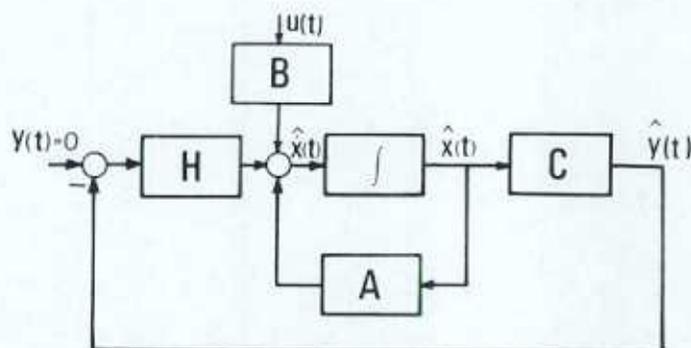


Figure 3. Closed-loop observer.

$\mu_i$  and  $v_i^T$  are the observer eigenvalue and left-eigenvector respectively. Observability of  $[A, C]$  guarantees the existence of  $H$  in (4). The observer has the structure of the Kalman filter. Combining the state-feedback and observer designs (Fig. 4) yields the unique compensator transfer-function matrix given by

$$K(s) = G(sI_{nn} - A + BG + HC)^{-1}H \quad (5)$$

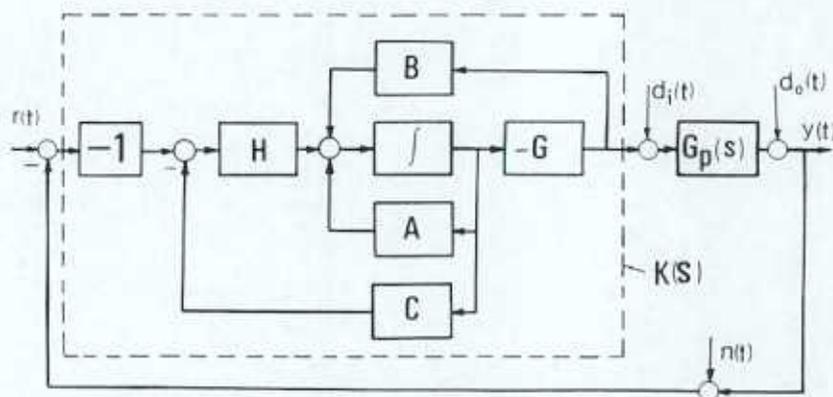


Figure 4. Closed-loop system.

The idea behind observer design is to find the steady-state filter gain,  $H$ , such that the loop transfer function  $K(s)G_p(s)$  in Fig. 1 maintains the same loop shape (for a bounded frequency range) that  $G(sI_{nn} - A)^{-1}B$  achieved via state-feedback design in the first stage. A technique for designing  $H$  to meet this criterion was offered by Doyle and Stein (1981). Since by this method  $K(s)G_p(s)$  preserves the loop shape achieved by  $G(sI_{nn} - A)^{-1}B$ , the final design in Fig. 1 meets the specifications that were already met by state-feedback design. (The title 'loop transfer recovery' comes from this idea.) The following lemma, which is proved by Doyle and Stein (1981) is central to the design of  $H$ .

#### Lemma

If  $H$  is chosen such that limit (6) is true as scalar  $\rho$  approaches infinity for any non-singular  $m \times m$   $W$ -matrix,

$$\frac{H(\rho)}{\rho} \rightarrow BW \quad (6)$$

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then  $K(s)$ , as given by (5), approaches pointwise (non-uniformly) towards (7):

$$G(sI_{nn} - A)^{-1}B[C(sI_{nn} - A)^{-1}B]^{-1} \quad (7)$$

and since

$$G_p(s) = C(sI_{nn} - A)^{-1}B \quad (8)$$

then  $K(s)G_p(s)$  will approach  $G(sI_{nn} - A)^{-1}B$  pointwise.†

The procedure requires only that  $H$  be stabilizing and have the asymptotic characteristic of (6). Doyle and Stein suggested one way to meet this requirement: a steady-state Kalman-filter gain (Kwakernak and Sivan 1972) with very small measurement-noise covariance. Now suppose we choose  $H$  with the following structure:

$$H = \rho BW \quad (9)$$

where  $W$  is any non-singular  $m \times m$  matrix and  $\rho$  is a scalar. It can be shown (by the definition of the limit) that the structure of  $H$  chosen in (9) satisfies the limit in (6) as  $\rho$  approaches infinity. In other words, as  $\rho$  approaches infinity,  $H \rightarrow \rho BW$  results in  $H/\rho \rightarrow BW$ . (The reverse is not true.) Since the structure of  $H$  given in (9) satisfies the limit in (6), then if  $H$  is chosen to be  $\rho BW$ ,  $K(s)G_p(s)$  will approach  $G(sI_{nn} - A)^{-1}B$  pointwise, as  $\rho$  approaches infinity. Note that the structure of  $H$  given by (9) does not necessarily yield a stable observer. We choose  $H$  to be  $\rho BW$  throughout this paper. The asymptotic finite eigenstructures of both forms given by (9) and (6) are the same, while the asymptotic infinite eigenstructures are usually different. The form in (9) usually yields an unstable infinite eigenstructure.

Although this paper is not an exposition of the properties of the transmission zeros of a plant, before stating the theorem, we will remind readers of some definitions and concepts about this matter. (For more information and properties of the transmission zeros, see Rosenbrock 1973, Desoer and Schulman 1974, Kouvaritakis and MacFarlane 1976.) The transmission zeros of a square plant are defined to be the set of complex numbers  $s_i$  that satisfy the following inequality:

$$\text{rank} \begin{bmatrix} s_i I_{nn} - A & B \\ C & 0_{mm} \end{bmatrix} < n + m$$

The necessary and sufficient condition for the truth of (10) is given by

$$\det \begin{bmatrix} s_i I_{nn} - A & B \\ B & 0_{mm} \end{bmatrix} = 0$$

Equation (11) yields  $j$  finite transmission zeros ( $j \leq n - m$ ). The remaining  $n - j$  transmission zeros are at infinity. For each finite transmission zero, there is one non-zero left null-vector  $[z_i^T \quad -w_i^T]$  (for  $i = 1, 2, \dots, j$ ) such that

$$[z_i^T \quad -w_i^T] \begin{bmatrix} s_i I_{nn} - A & B \\ C & 0_{mm} \end{bmatrix} = 0_{n+m}^T$$

† The pointwise approach indicates the approximate equality of  $K(s)G_p(s)$  and  $G(sI_{nn} - A)^{-1}B$  for some bounded frequency range.

where

$$[z_i^T \quad -w_i^T] \neq 0_{n+m}^T$$

$w_i^T$  is an  $m \times 1$  vector and  $z_i^T$  is an  $n \times 1$  vector.  $z_i^T$  is called left zero-direction of the transmission zeros of the plant. A similar definition for the transmission zeros of a square plant is given by Kwakernak and Sivan 1972; all complex numbers that are roots of  $\Psi(s)$  in the equation:

$$\det G_p(s) = \frac{\Psi(s)}{\Phi_o(s)} \quad (13)$$

are transmission zeros of the plant.  $\Phi_o(s)$  is the  $n$ th-order open-loop characteristic equation. The maximum order of  $\Psi(s)$  is  $j$ . All transmission zeros of the plant, including the ones that are equal to the eigenvalues of the plant (which may even be the input-decoupling and/or output-decoupling zeros of the system (MacFarlane and Karcnias 1976), are roots of  $\Psi(s)$  and also satisfy (10) and (11). The equivalence of (13) and (11) can be shown by careful use of Schur's equality (Kailath 1980).

### 3. Asymptotic eigenstructure properties of the LTR method

We will now explore some eigenstructure properties for LTR when the observer gain satisfies (9). Knowing the eigenstructure properties of the compensator, we will develop a method for designing  $H$  via eigenstructure assignment of the observer. The following theorem gives the eigenstructure properties of the observer when  $H$  is chosen according to (9). Part (i) of the theorem is proved differently by Davison and Wang (1974), and can also be considered to be a special result of the multivariable root locus given by Owens (1977), Shaked and Kouvaritakis (1977), Kouvaritakis and MacFarlane (1976) and Kouvaritakis and Shaked (1976). The second part of the theorem is the result we will use in the design process.

#### Theorem

Consider the square linear observer in Fig. 3:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + He(t) + Bu(t) \quad (14)$$

$$e(t) = -C\hat{x}(t) + y(t) \quad (15)$$

$$\hat{x}(t) \in R^n, \quad u(t), y(t) \in R^m$$

with

$$\text{rank}(B) = \text{rank}(C) = m$$

Then if  $H$  is chosen so that

$$H = \rho BW \quad (16)$$

where  $W$  is any non-singular square matrix and  $\rho$  is a scalar approaching  $\infty$ , then the following statements are true.

- (i) The finite closed-loop eigenvalues of  $A - HC$ ,  $\mu_i$ , approach finite transmission zeros of the plant,  $s_i$ . If the linear plant  $[A, B, C]$  has  $j$  finite transmission zeros ( $j \leq n - m$ ) then  $A - HC$  will have  $j$  finite eigenvalues. The remaining closed-loop eigenvalues approach infinity at any angle.
- (ii) The left closed-loop eigenvector  $v_i^T$  ( $i = 1, 2, \dots, j$ ) associated with the finite

closed-loop eigenvalue  $\mu_i$  approaches  $z_i^T$ , which satisfies the following:

$$\begin{aligned} [z_i^T \quad -w_i^T] \begin{bmatrix} s_i I_{nn} - A & B \\ C & 0_{mm} \end{bmatrix} &= 0_{n+m}^T \\ [z_i^T \quad -w_i^T] &\neq 0_{n+m}^T \end{aligned} \quad (17)$$

$w_i^T$  and  $z_i^T$  are  $m \times 1$  and  $n \times 1$  vectors respectively. If  $s_i$  is not equal to any eigenvalue of  $A$  then  $z_i^T$  can be computed from (17), and the following expression for  $v_i^T$  ( $i = 1, 2, \dots, j$ ) can be obtained:

$$v_i^T = w_i^T [C(s_i I_{nn} - A)^{-1}] \quad (18)$$

where  $w_i^T$  ( $i = 1, 2, \dots, j$ ) can be calculated from

$$w_i^T [C(s_i I_{nn} - A)^{-1} B] = 0_m^T \quad (19)$$

where

$$w_i^T \neq 0_m^T$$

### Interpretation

This theorem identifies the asymptotic locations of finite closed-loop eigenvalues and left eigenvectors of the observer. As  $\rho$  approaches a large number,  $j$  (for  $j \leq n - m$ ) closed-loop eigenvalues will approach finite transmission zeros of the plant, and  $n - j$  closed-loop eigenvalues will approach infinity at any angle. Since conventional practice in complex-variable work is to regard a function as having an equal number of poles and zeros when the zeros at infinity are included, one can claim that all closed-loop eigenvalues approach the transmission zeros of the plant. Equation (17) states that  $[v_i^T \quad -w_i^T]$  is confined in the left null-space of the given matrix in (17) as  $\rho$  approaches infinity. In other words, the left null-space of the matrix given in (17) assigns a subspace for limiting location of  $[v_i^T \quad -w_i^T]$  when  $\rho$  approaches infinity. If  $s_i$  is not equal to any eigenvalues of  $A$ , the limiting location of  $v_i^T$  can be interpreted differently. Equation (18) states that the left-eigenvector  $v_i^T$  is confined to a subspace spanned by the rows of  $C(s_i I_{nn} - A)^{-1}$  if  $s_i$  is not equal to any eigenvalues of  $A$ . This subspace is of dimension equal to the rank of  $C$ . Therefore the number of independent output variables determines how large the subspace corresponding to the left closed-loop eigenvector can be. The orientation of each subspace associated with each left closed-loop eigenvector  $v_i^T$  depends on the open-loop dynamics of the system  $[A, C]$  and the closed-loop observer eigenvalue  $\mu_i$ . Construction of the left closed-loop eigenvectors in their allowable  $m$ -dimensional subspace in  $C^n$  is the exact freedom that is offered by observer design beyond pole placement (Klein and Moore 1977, Porter and D'Azzo 1978 a, b, Harvey and Stein 1978). This theorem identifies the asymptotic  $m$ -dimensional subspace in  $C^n$  that confines the left closed-loop eigenvector  $v_i^T$ . The choice of  $w_i^T$  in (18) allows the designer to construct each  $n$ -dimensional left closed-loop eigenvector in its allowable  $m$ -dimensional subspace. As  $\rho$  approaches a large number then  $w_i^T$  approaches the left null-vector of  $G_p(s_i)$  in (19); consequently, each left closed-loop eigenvector  $v_i^T$  approaches a final value in its allowable subspace given by (18).

### Proof

#### Part (i)

$H$  is chosen according to (16). The block diagram of the closed-loop observer is shown in Fig. 5. The loop transfer function at the plant output is given by (20).

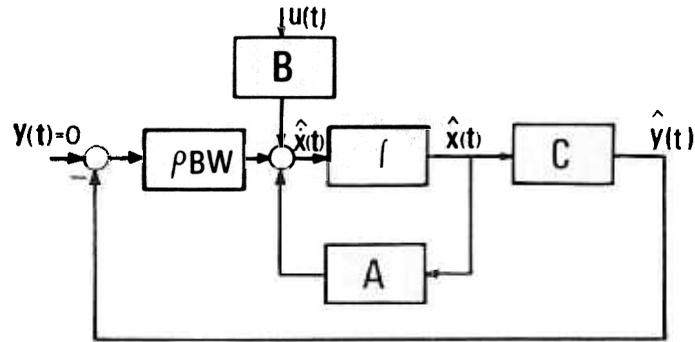


Figure 5. Closed-loop observer configuration.

$$C(sI_{nn} - A)^{-1}\rho BW \quad (20)$$

The following equation relates the open-loop and closed-loop characteristic equations (MacFarlane 1972, Rosenbrock 1974):

$$\det [I_{nn} + C(sI_{nn} - A)^{-1}\rho BW] = \frac{\Phi_{cl}(s)}{\Phi_{ol}(s)} \quad (21)$$

where  $\Phi_{cl}(s) \equiv$  closed-loop characteristic equation of the system in Fig. 5,  $\Phi_{ol}(s) \equiv$  open-loop characteristic equation of the system in Fig. 5. From matrix theory, the following is true (Kailath 1980):

$$\begin{aligned} \det [I_{nn} + C(sI_{nn} - A)^{-1}\rho BW] \\ = I_{nn} + \text{trace} [C(sI_{nn} - A)^{-1}\rho BW] + \dots + \det [C(sI_{nn} - A)^{-1}\rho BW] \end{aligned} \quad (22)$$

As  $\rho$  approaches  $\infty$  the last term of (22) grows faster than the other terms. Therefore the following approximation is true:

$$\det [I_{nn} + C(sI_{nn} - A)^{-1}\rho BW] \approx \det [C(sI_{nn} - A)^{-1}\rho BW] \quad (23)$$

Considering approximation (23), (21) can be written as

$$\det [C(sI_{nn} - A)^{-1}\rho BW] \approx \frac{\Phi_{cl}(s)}{\Phi_{ol}(s)} \quad (24)$$

or equivalently

$$\det [G_p(s)] \det [\rho W] \approx \frac{\Phi_{cl}(s)}{\Phi_{ol}(s)} \quad (25)$$

Since  $\det [\rho W] \neq 0$ , comparing (13) and (25) shows that the roots of  $\Psi(s)$  and  $\Phi_{cl}(s)$  are the same. In other words,  $\Phi_{cl}(s)$  produces all the transmission zeros of the plant, including those that are equal to the eigenvalues of  $A$ .

#### Part (ii)

When  $H$  approaches its asymptotic value, the eigenvalues of  $A - HC$  can no longer be moved via matrix  $C$ . This is true because the eigenvalues of  $A - HC$  are at their limiting locations (i.e. transmission zeros of the plant). Therefore  $[A - HC, B, C]$  must have unobservable or uncontrollable modes. Since  $[A - HC, C]$  is an observable

pair and  $H$  is expressed as  $\rho BW$ ,  $[A - HC, B]$  must be an uncontrollable pair. Since  $[A - HC, B]$  is an uncontrollable pair, the following equations are true (MacFarlane and Karcnias 1976):

$$v_i^T(\mu_i I_{nn} - A + HC) = 0_n^T, \quad i = 1, 2, \dots, j \quad (26)$$

$$v_i^T B = 0_m^T \quad (27)$$

$\mu_i$  is the closed-loop observer eigenvalue and  $v_i^T$  is the corresponding left-eigenvector. Equation (27) states that the left closed-loop eigenvector  $v_i^T$  from (26) is in the left null-space of  $B$  and cannot be affected by the input. Each closed-loop eigenvector  $v_i^T$  (for  $i = 1, 2, \dots, j$ ) can be expressed by

$$v_i^T(\mu_i I_{nn} - A) - w_i^T C = 0_n^T \quad (28)$$

where

$$w_i^T = -v_i^T H$$

Combining (28) and (27) yields (30) (note that  $s_i = \mu_i$ ):

$$\begin{bmatrix} v_i^T & -w_i^T \end{bmatrix} \begin{bmatrix} s_i I_{nn} - A & B \\ C & 0_{mm} \end{bmatrix} = 0_{n+m}^T \quad (30)$$

where

$$\begin{bmatrix} v_i^T & -w_i^T \end{bmatrix} \neq 0_{n+m}^T \quad \text{for } i = 1, 2, \dots, j$$

If  $s_i$  is not equal to any eigenvalue of  $A$  then from (30) we can find an expression for the left closed-loop eigenvector of  $A$ :

$$v_i^T = w_i^T C (s_i I_{nn} - A)^{-1}, \quad i = 1, 2, \dots, j \quad (31)$$

where  $w_i^T$  can be computed from

$$\begin{aligned} w_i^T [C (s_i I_{nn} - A)^{-1} B] &= 0_m^T, \quad i = 1, 2, \dots, j \\ w_i^T &\neq 0_m^T \end{aligned} \quad (32)$$

Equation (31) shows that the left-eigenvectors achievable for the closed-loop observer are confined to the  $m$ -dimensional subspaces determined by their associated eigenvalues and open-loop dynamics  $[A, C]$ .  $\square$

#### Comment

As  $\rho$  approaches  $\infty$  the  $j$  eigenvalues of  $A - HC$  cancel out the  $j$  finite transmission zeros of the plant. A cancellation of an equal-valued closed-loop eigenvalue of the system with a transmission zero happens if the left closed-loop eigenvector of the system is equal to the left zero-direction  $z_i^T$  associated with the transmission zero in (17). By cancelling we mean they will not appear as poles in the closed-loop transfer-function matrix  $C[sI_{nn} - A + HC]^{-1}B$ . The transmission zeros of  $[A, B, C]$  are the same as those of  $[A - HC, B, C]$ , because transmission zeros do not change under feedback. As  $\rho$  approaches infinity the transmission zeros of  $[A - HC, B, C]$  turn into input-decoupling zeros, because the system of  $[A - HC, B, C]$  is not controllable at these modes (MacFarlane and Karcnias 1976).

#### Corollary 1

The finite transmission zeros of  $K(s)$  are the same as the finite transmission zeros of  $G(sI_{nn} - A)^{-1}B$ .

*Proof*

The transmission zeros of  $G(sI_{nn} - A)^{-1}B$  are the complex values  $\sigma_i$  that satisfy the following inequality:

$$\text{rank} \begin{bmatrix} \sigma_i I_{nn} - A & B \\ G & 0_{mm} \end{bmatrix} < n + m \quad (33)$$

Postmultiplying the matrix in (33) the non-singular matrix

$$\begin{bmatrix} I_{nn} & 0_{nm} \\ G + \rho WC & \rho W \end{bmatrix} \quad (34)$$

will result in the following inequality for the transmission zeros of  $G(sI_{nn} - A)^{-1}B$ :

$$\text{rank} \begin{bmatrix} \sigma_i I_{nn} - A + BG + \rho BWC & \rho BW \\ G & 0_{mm} \end{bmatrix} < n + m$$

Substituting  $H$  for  $\rho BW$  in (35) results in

$$\text{rank} \begin{bmatrix} \sigma_i I_{nn} - A + BG + HC & H \\ G & 0_{mm} \end{bmatrix} < n + m \quad (36)$$

The complex number  $\sigma_i$  that satisfies (36) is a transmission zero of  $K(s)$  as given by (5). Therefore  $K(s)$  and  $G(sI_{nn} - A)^{-1}B$  have equal transmission zeros. If  $G(sI_{nn} - A)^{-1}B$  does not have any finite transmission zeros then  $K(s)$  will not have any finite transmission zeros.  $\square$

*Corollary 2*

If  $\rho$  approaches  $\infty$ , then all the eigenvalues of the compensator  $K(s)$  will approach the transmission zeros (including the ones at infinity) of the plant, and the left-eigenvectors of  $A - BG - HC$ ,  $x_i^T$ , will approach  $z_i^T$ , where  $z_i^T$  and  $s_i$  ( $i = 1, 2, \dots, j$ ) satisfy

$$\begin{aligned} [z_i^T \quad -w_i^T] \begin{bmatrix} s_i I_{nn} - A & B \\ C & 0_{mm} \end{bmatrix} &= 0_{n+m}^T \\ [z_i^T \quad -w_i^T] &\neq 0_{m+n}^T \end{aligned} \quad (37)$$

In other words, the eigenvalues of the compensator cancel out the transmission zeros of the plant.

*Proof*

The transmission zeros of the plant are the set of complex numbers  $s_i$  that satisfy (37). Postmultiplying (37) by the non-singular matrix

$$\begin{bmatrix} I_{nn} & 0_{nm} \\ G & I_{mm} \end{bmatrix}$$

will yield the following equation, which can then be solved to find the finite

transmission zeros of the plant:

$$[z_i^T \quad -w_i^T] \begin{bmatrix} s_i I_{nn} - A + BG & B \\ C & 0_{mm} \end{bmatrix} = 0_{n+m}^T \quad (39)$$

$$[z_i^T \quad -w_i^T] \neq 0_{n+m}^T \quad \text{for } i = 1, 2, \dots, j$$

We apply the result of the theorem to system  $[A - BG, B, C]$ . According to part (i) of the theorem, if  $H = \rho BW$ , then as  $\rho$  approaches  $\infty$ , the eigenvalues of  $A - BG - HC$  will approach the transmission zeros of  $[A - BG, B, C]$  computed from (39). These are also the transmission zeros of the plant given by (37).

According to part (ii) of the theorem, the left closed-loop eigenvectors  $x_i^T$  of the compensator given by

$$x_i^T (\mu_i I_{nn} - A + BG + HC) = 0_n^T, \quad i = 1, 2, \dots, j \quad (40)$$

$$x_i^T \neq 0_n^T$$

approaches  $z_i^T$  given by (39) or (37). □

#### 4. Comments

According to Corollary 2, as  $\rho$  approaches  $\infty$ , the eigenvalues of  $K(s)$  will cancel out the transmission zeros of the plant. According to Corollary 1, as  $\rho$  approaches  $\infty$  the transmission zeros of  $K(s)$  will approach the transmission zeros of  $G(sI_{nn} - A)^{-1}B$ . Since the number of transmission zeros of two cascaded systems ( $K(s)$  and  $G_p(s)$ ) is the sum of the number of transmission zeros of both systems, the transmission zeros of  $K(s)G_p(s)$  are the same as the transmission zeros of  $G(sI_{nn} - A)^{-1}B$ . Similar arguments can be given for the poles of  $K(s)G_p(s)$ . The poles of  $K(s)$  cancel out the transmission zeros of the plant; therefore the poles of  $K(s)G_p(s)$  will be the same as poles of  $G(sI_{nn} - A)^{-1}B$ . This argument does not prove the equality of  $G(sI_{nn} - A)^{-1}B$  and  $K(s)G_p(s)$  as  $\rho$  approaches  $\infty$ . Proof of the pointwise equality of  $K(s)G_p(s)$  and  $G(sI_{nn} - A)^{-1}B$  is best shown by Doyle and Stein (1981). The above comment concerning pole-zero cancellation explains the eigenstructure mechanism for LTR. Since pole placement and eigenvector construction in the allowable subspace prescribes a unique value for  $H$ , we plan to design the observer gain for the LTR via pole placement and left-eigenvector construction.

The asymptotic finite eigenstructure for  $H$  in both (6) and (9) are the same, but the asymptotic infinite eigenstructures are usually different. The form of  $H$  given by (9) is rarely stabilizing. Since both forms guarantee the pointwise approach of  $K(s)G_p(s)$  to  $G(sI_{nn} - A)^{-1}B$ , it can be deduced that the pointwise approach of  $K(s)G_p(s)$  to  $G(sI_{nn} - A)^{-1}B$  occurs whenever the asymptotic finite eigenstructure is the same as that given by the theorem. Hence combining any such finite eigenstructure with any stable infinite eigenstructure will result in the approach of  $K(s)G_p(s)$  to  $G(sI_{nn} - A)^{-1}B$  in a stable sense.

Difficulty in using LTR will arise if the plant has some right half-plane zeros (non-minimum-phase plant). In our proposed procedure for LTR, one should place the eigenvalues of  $A - HC$  at the transmission zeros of the plant. If the plant is non-minimum phase, one would place some eigenvalues of  $A - HC$  on the right half-plane. The closed-loop system will not be stable if any eigenvalues of  $A - HC$  are on the right half-plane. According to the separation theorem, the eigenvalues of  $A - HC$  are also the eigenvalues of the closed-loop system. Therefore the sufficient condition for LTR

and the stability of the closed-loop system is that the plant be minimum-phase. If the plant is non-minimum-phase, one should consider the mirror images of the right half-plane zeros as target locations for eigenvalues of  $A - HC$ . In such cases, loop transfer recovery is not guaranteed, but the closed-loop system will be stable.

### 5. Design method

For observer design, we place  $j$  finite eigenvalues of  $A - HC$  at finite transmission zeros of the plant. The left closed-loop eigenvector  $v_i^T$  associated with the finite modes must be constructed such that  $[v_i^T \quad -w_i^T]$  is in the left null-space of the matrix given by (17). The remaining  $n - j$  closed-loop eigenvalues should be placed far in the left half-plane. Note that the farther the  $n - j$  infinite eigenvalues of  $A - HC$  are located from the imaginary axis, the closer  $K(s)G_p(s)$  will be to  $G(sI_{nn} - A)^{-1}B$  as shown in the example. The left closed-loop eigenvectors associated with the infinite modes can be computed via

$$v_i^T = w_i^T C (\mu_i I_{nn} - A)^{-1} \quad = j + 1, j + 2, \quad (41)$$

where

$$w_i^T = -v_i^T H \quad (42)$$

The following steps will lead a designer toward observer design for the recovery procedure.

#### Step 1

Use (17) to compute the  $j$  target locations of the complex finite eigenvalues of the observer,  $s_i$ , and  $j$  left null-vectors of  $[z_i^T \quad -w_i^T]$ .  $\mu_i$  must be selected to be equal to  $s_i$ . The left closed-loop eigenvector of the observer,  $v_i^T$ , must be selected to be equal to  $z_i^T$ . If  $s_i$  is not equal to any eigenvalue of  $A$ , use (18) and (19) to compute the  $j$  left closed-loop eigenvectors  $v_i^T$  and  $w_i^T$ .  $w_i^T$  identifies the location of the left closed-loop eigenvector in its allowable subspace. This step terminates the construction of the finite eigenstructure of the observer.

#### Step 2

Place the remaining  $n - j$  eigenvalues of  $A - HC$  at locations farther than the finite transmission zeros of the plant. Use (41) to achieve  $n - j$  values for  $v_i^T$ . The  $w_i^T$  for infinite modes are arbitrary and have little importance because their corresponding eigenvalues are selected far in the left-half complex plane.

#### Step 3

Since

$$v_i^T H = -w_i^T, \quad = 1, 2, \dots, n \quad (43)$$

$$\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} H = - \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix} \quad (44)$$

Compute  $H$  as

$$H = - \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix} \quad (45)$$

The independence of the  $n$  left closed-loop eigenvectors,  $v_i^T$  is a necessary condition to use eigenstructure assignment for LTR. If the left closed-loop eigenvectors are not independent, our approach fails and one must use Doyle and Stein's approach to recover the loop transfer function. The dependency of the left eigenvectors might arise if multiple finite transmission zeros result in (17). If degeneracy of the matrix in (17) is equal to the multiplicity of a transmission zero then the existence of  $n$  independent finite left closed-loop eigenvectors is guaranteed.

## 6. Example

Consider the following example:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 76 & -105 \\ 105 & 280 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

Suppose we are given  $G$  such that the closed-loop poles are at  $-19.35$ ,  $-1.76$ ,  $-5.57$  and  $-6.12$ :

$$G = \begin{bmatrix} 4.7234 & 3.4265 & 0.9923 & 0.6631 \\ 1.1497 & 0.8579 & 0.2633 & 0.1952 \end{bmatrix}$$

Using (19), the finite transmission zeros  $s_i$  and the associated left null-vector directions  $w_i^T$  can be computed.  $\mu_1$  and  $\mu_2$  are selected to be equal to  $s_1$  and  $s_2$ :

$$\mu_1 = -1, \quad \mu_2 = -0.25, \quad w_1^T = [1 \ 0], \quad w_2^T = [0 \ 1]$$

Using (18), the left closed-loop eigenvector associated with the finite modes can be computed as

$$v_1^T = [-1 \ 0 \ 0 \ 0] \\ v_2^T = [0 \ -4 \ 0 \ 0]$$

We place the other two eigenvalues of  $A - HC$  in the left half-plane as far as possible. The directions of  $w_3^T$  and  $w_4^T$  do not matter because the associated eigenvalues are far away. Figure 6 shows that the farther away the two infinite eigenvalues of  $A - HC$  are from the imaginary axis, the closer  $K(s)G_P(s)$  will be to  $G(sI_m - A)^{-1}B$ . Assuming

$$\mu_3 = -30, \quad \mu_4 = -36, \quad w_3^T = [1 \ 0], \quad w_4^T = [0 \ 1]$$

and using (41), the left-eigenvectors associated with infinite modes can be computed as

$$v_3^T = [-0.0333 \ 0 \ -0.0322 \ 0] \\ v_4^T = [0 \ -0.0278 \ 0 \ -0.1103]$$

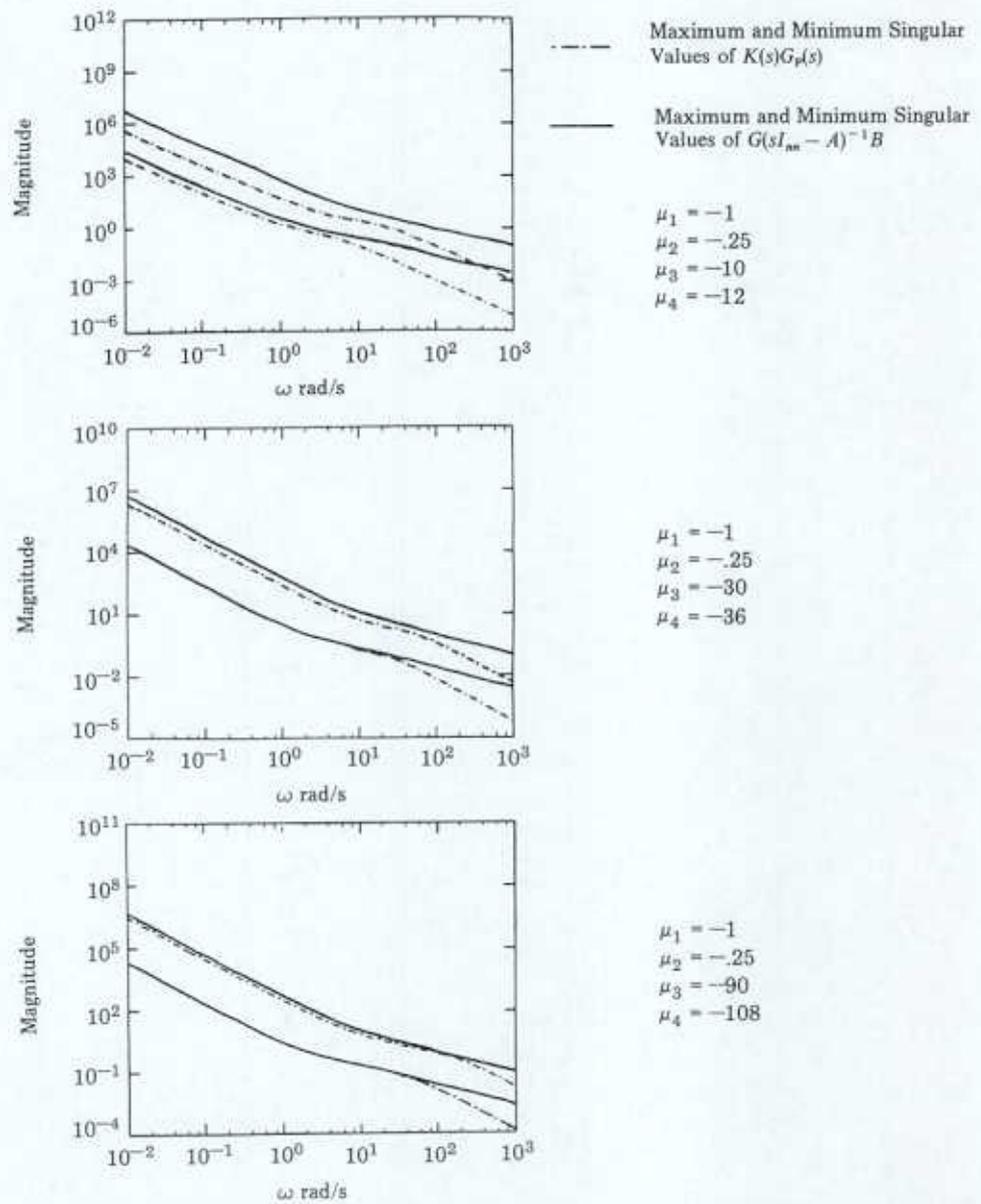


Figure 6. Maximum and minimum singular values of  $K(s)G_p(s)$  and  $G(sI_m - A)^{-1}B$ .

Using (45),  $H$  can be computed as

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \\ 30 & 0 \\ 0 & 0.9 \end{bmatrix}$$

The finite transmission zeros of  $G(sI_{nn} - A)^{-1}B$  are located at  $-4.3270$  and  $-1.3675$ . The Table shows that the transmission zeros of  $K(s)$  approach the transmission zeros of  $G(sI_{nn} - A)^{-1}B$  as  $\mu_3$  and  $\mu_4$  move farther into the left-half complex plane (Corollary 1). The Table also shows that the farther  $\mu_3$  and  $\mu_4$  are from the imaginary axis, the closer the eigenvalues of  $K(s)$  will be to the transmission zeros of the plant (Corollary 2).

Closed-loop eigenvalues	Transmission zeros of $K(s)$	Eigenvalues of $K(s)$
$\mu_1 = -1$	$-2.8097$	$-30.5672$
$\mu_2 = -0.25$	$-1.2662$	$-24.2387$
$\mu_3 = -10$	$-\infty$	$-1$
$\mu_4 = -12$	$-\infty$	$-0.25$
$\mu_1 = -1$	$-3.6734$	$-49.4030 + 10.4478i$
$\mu_2 = -0.25$	$-1.3311$	$-49.4030 - 10.4478i$
$\mu_3 = -30$	$-\infty$	$-1$
$\mu_4 = -36$	$-\infty$	$-0$
$\mu_1 = -1$	$-4.0855$	$-115.40 + 20.46i$
$\mu_2 = -0.25$	$-1.3551$	$-115.40 - 20.46i$
$\mu_3 = -90$	$-\infty$	$-1$
$\mu_4 = -108$	$-\infty$	$-0.25$

Poles and zeros of  $K(s)$ .

## 7. Conclusion

A key step in the recent work on the synthesis of model-based feedback compensators for multivariable systems is the selection of the observer gain. The observer gain must be selected so that the final loop-transfer function  $K(s)G_p(s)$  in Fig. 1 is the same as the state-feedback loop transfer function  $G(sI_{nn} - A)^{-1}B$  (shown in Fig. 2) for some bounded frequency range. In LTR the eigenvalues of the compensator  $K(s)$  cancel the transmission zeros of the plant. It is also true that the compensator  $K(s)$  will share the same transmission zeros as  $G(sI_{nn} - A)^{-1}B$ . By exploring the eigenstructure of the model-based compensator when loop transfer recovery takes place, we provide an alternative design procedure which eliminates the need for the Kalman-filter mechanism via direct assignment of the eigenvalues and left-eigenvectors of the observers. The sufficient condition for LTR and the stability of the closed-loop system is that the plant be minimum-phase. The limitation of this method might arise when the plant has multiple finite transmission zeros and  $n$  left independent closed-loop eigenvectors cannot be constructed.

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