On the Robot Compliant Motion Control

H. Kazerooni
Mechanical Engineering Department,
University of Minnesota,
Minneapolis, Minn. 55455

1 Introduction

Most assembly operations and manufacturing tasks require mechanical interactions with the environment or with the object being manipulated, along with "fast" motion in unconstrained space. In constrained maneuvers, the interaction force must be accommodated rather than resisted. Two methods have been suggested for development of compliant motion. The first approach is aimed at controlling force and position in a nonconflicting way [10, 11, 12, 18]. In this method, force is commanded along those directions constrained by the environment, while position is commanded along those directions in which the manipulator is unconstrained and free to move. The second approach is focused on developing a relationship between the interaction force and the manipulator position [1, 4, 5, 13]. By controlling the manipulator position, specifying its relationship with the interaction force, a designer can ensure that the manipulator will be able to maneuver in a constrained space while maintaining appropriate contact force. This paper describes an analysis on the control and stability of the robot in constrained maneuvers when the second method is employed to control the robot compliancy.

We start with modeling the robot and the environment with unstructured dynamic models. To arrive at a general stability criterion, we avoid using structured dynamic models such as first or second order transfer functions as general representations of the dynamic behavior of the components of the robot (such as actuators). Using unstructured models for the robot and environment, we analyze the stability of the robot and environment via the Small Gain Theorem and Nyquist Criterion. We show that the stability criterion achieved via the Nyquist method is a subclass of the condition given by the Small Gain Theorem. For a particular application, one can replace the unstructured dynamic models with known models and then a tighter condition can be achieved. The stability criterion reveals that there must be some initial compliancy either in the robot or in the environment. The initial compliancy in the robot can be obtained by a passive compliant element such as an RCC (Remote Center Compliance) or compliancy within the positioning feedback. Practitioners always observed that the system of a robot and a stiff environment can always be stabilized when a compliant element (e.g., piece of rubber or an RCC) is installed between the robot and environment. The stability criterion also shows that no compensator can be found to stabilize the interaction of the ideal positioning system (very rigid tracking robot) with an infinitely rigid environment. In this case the robot and environment both resemble ideal sources of flow (defined in bond graph theory) and they do not physically complement each other. A fast, light-weight, active end-effector (a miniature robot) which can be attached to the end-point of a commercial robot manipulator has been designed and built to experimentally verify the control method.

2 Dynamic Model of the Robot With Positioning Controllers

In this section, a general approach will be developed to describe the dynamic behavior of a large class of industrial and research robot manipulators having positioning (tracking) controllers. The fact that most industrial manipulators already have some kind of positioning controller is the motivation behind our approach. Also, a number of methodologies exist for the development of robust positioning controllers for direct and nondirect robot manipulators [14, 17].

In general, the end-point position of a robot manipulator that has a positioning controller is a dynamic function of its
input trajectory vector, \( e \), and the external force, \( d \). Let \( G \) and \( S \) be two functions that show the robot end-point position in a global coordinate frame, \( y \), is a function of the input trajectory, \( e \), and the external force, \( d \). \( (d \) is measured in the global coordinate frame also.)

\[
y = G(e) + S(d) \tag{1}
\]

The motion of the robot end-point in response to imposed forces, \( d \), is caused by either structural compliance in the robot or by the compliance of the positioning controller. Robot manipulators with tracking controllers are not infinitesimally stiff in response to external forces (also called disturbances). Even though the positioning controllers of robots are usually designed to follow the trajectory commands and reject disturbances, the robot end-point will move somewhat in response to imposed forces on it. \( S \) is called the sensitivity function and it maps the external forces to the robot end-point position. For a robot with a “good” positioning controller, \( S \) is a mapping of small gain. \( \) The maximum singular value of a matrix, \( \alpha_{\text{max}}(A) \), is defined as:

\[
\alpha_{\text{max}}(A) = \max \{ |z| : \| A z \| = \| z \|, \| z \| \leq 1 \}
\]

where \( z \) is a nonzero vector and \( \| \cdot \| \) denotes the Euclidean norm of a vector or a scalar.

3 Dynamic Behavior of the Environment

The environment can be very “soft” or very “stiff.” We do not assume that linear superposition (in equation (1)) holds for the external forces on the RCC. Obtained results to cover the case when \( G(e) \) and \( S(d) \) do not superimpose. The nonlinear analysis do not depend on this assumption and one can extend the results to the nonlinear case.

A similar modeling method can be given for the analysis of the linearly treated robots. The transfer function matrices, \( G \) and \( S \) in equation (2) are defined to describe the dynamic behavior of a linearly treated robot manipulator with positioning controller.

\[
y(jw) = G(jw)e(jw) + S(jw)d(jw) \tag{2}
\]

In equation (2), \( S \) is called the sensitivity transfer function and it maps the external forces to the end-point position. \( G(jw) \) is the closed-loop transfer function matrix that maps the input trajectory vector, \( e \), to the robot end-point position, \( y \). For a robot with a “good” positioning controller, \( S(jw) \) is “small” in the singular value sense, while \( G(jw) \) is approximately a unity matrix.

### Nomenclature

- \( A = \) the closed-loop mapping from \( r \) to \( e \) in Fig. 4
- \( d = \) vector of the external force on the robot end-point
- \( e = \) input trajectory vector
- \( E = \) environment dynamics
- \( f = \) vector of the contact force,
  \( U_1, f_2, \ldots, f_n \)
- \( f_m = \) the limiting value of the contact force for infinitely rigid environment
- \( G = \) robot dynamics with positioning controller
- \( H = \) compensator
  (operating on the contact force, \( f \))
- \( I_n = \) identity matrix
- \( j_i = \) moment of inertia of each link relative to the end-point of the link
- \( j = \) complex number notation \( \sqrt{-1} \)
- \( J_e = \) Jacobian
- \( l_i, m_i = \) length and mass of each link
- \( M_0 = \) inertia matrix
- \( n = \) degrees of freedom of the robot \( n \leq 6 \)
- \( S = \) robot manipulator sensitivity (1/stiffness)
- \( T = \) positive scalar
- \( V = \) the forward loop map- ping from \( r \) to \( f \) in Fig. 4
- \( x = \) vector of the environment deflection
- \( \epsilon_0, \epsilon_1, \mu, \gamma = \) positive scalars
- \( \omega_0 = \) frequency range of operation (bandwidth)
- \( \alpha, \beta, \rho = \) positive scalars
- \( \sigma = \) small perturbation of \( \theta_0 \) in the neighborhood of \( \theta_0 = 90 \) deg
- \( \delta = \) end-point deflection in \( y_n \)-direction
- \( \delta_x, \delta_y = \) end-point deflection in the direction normal and tangential to the part
- \( \omega_d = \) dynamic manipulability
- \( e = \) vector of the environ ment position before contact
- \( \theta = \) vector of the joint angles of the robot
- \( \tau = \) vector of the robot torques
- \( \| \cdot \| \) denotes the Euclidean norm of a vector or a scalar.
not restrain ourselves to any geometry or to any structure. If one point on the environment is displaced as vector of \( x \), with force vector, \( f \), then the dynamic behavior of the environment is given by equation (3).

\[
    f = E(x)
\]

(3)

If \( x_0 \) is the initial location of the point of contact on the environment before deformation occurs then, \( x = y - x_0 \). \( E \) is assumed to be stable in \( L_p \)-sense; \( E L_p E^* - L_p \) and \( \| E(x) \|_p \leq \|x\|_p + \beta \). Confining equation (3) to cover the linearly treated environment, equation (4) represents the dynamic behavior of the environment.

\[
    f(\omega)=E(\omega)x(\omega)
\]

(4)

\( E(\omega) \) is a transfer function matrix that maps the displacement vector, \( x \), to the contact force, \( f \). Matrix \( E \) is a \( n \times n \) transfer function matrix. \( E \) is a singular matrix when the robot interacts with the environment in only some directions. For example, in grinding a surface, the robot is constrained by the environment in the direction normal to the surface only. Readers can be convinced of the truth of equation (4) by analyzing the relationship of the force and displacement of a mass, spring and damper as a simple model of the environment. \( E \) resembles the impedance of a system. References [4 and 5] represent \( (Js^2 + Ds + K) \) for \( E \) where \( J \), \( D \), and \( K \) are symmetric matrices and \( s = j\omega \) (4).

4 Nonlinear Dynamic Behavior of the Robot and Environment

Suppose a manipulator with dynamic equation (1) is in contact with an environment given by equation (3); then \( f = -d \). Figure 2 shows the dynamics of the robot manipulator and environment when they are in contact with each other. Note that in some applications, the robot will have only unidirectional force on the environment. For example, in grinding a surface, the robot can only push the surface. If one considers positive \( f \) for "pushing" and negative \( f \) for "pulling," then in this class of manipulation, the robot manipulator and the environment are in contact with each other only along those directions where \( f_i > 0 \) for \( i = 1, \ldots, n \). In some applications such as screwing a bolt, the interaction force can be positive and negative. This means the robot can have clockwise and counterclockwise interaction torque. The nonlinear discriminator block diagram in Fig. 2 is drawn with dashed-line to illustrate the above concept.

Using equations (1) and (3), equations (5) and (6) represent the entire dynamic behavior of the robot and environment taken as a whole.

\[
    y = G(e) + S(-f)
\]

(5)

\[
    f = E(x) \quad \text{where} \quad x = y - x_0
\]

(6)

If all the operators in Fig. 2 are considered linear transfer function matrices, equations (7) and (8) can be obtained to represent the end point position and the contact force when \( x_0 = 0 \).

\[
    y = (I_n + SE) G e
\]

(7)

\[
    f = E(I_n + SE) G e
\]

(8)

To simplify the block diagram of Fig. 2, we introduce a mapping from \( e \) to \( f \).

\[
    f = V(e)
\]

(9)

If all the operators in Fig. 2 are transfer function matrices, then \( V = E(I_n + SE)^{-1} G \). \( V \) is assumed to be a stable operator in \( L_p \)-sense; therefore \( V L_p V^* = L_p \) and also \( \|V(e)\|_p \leq \|e\|_p + \beta \). With this assumption, we basically claim that a robot with stable tracking controller remains stable when it is in contact with an environment. Note that one can still define \( V \) without assuming the superposition of effects of \( e \) and \( d \) in equation (5) (or equation (1)).

5 The Architecture of the Closed-Loop System

We propose the architecture of Fig. 3 to develop compliance for the robot. The compensator, \( H \), is considered to operate on the contact force, \( f \). The compensator output signal is being subtracted from the input command vector, \( r \), resulting in the input trajectory vector, \( e \), for the robot manipulator. The readers should be reminded that the robot in Fig. 3 can be considered a weak tracking robot (open loop robot without any feedback on the position and velocity) when the gain of \( S \) is a large number.

There are two feedback loops in the system; the inner loop (which is the natural feedback loop), is the same as the one shown in Fig. 2. This loop shows how the contact force affects the robot in a natural way when the robot is in contact with the environment. The outer feedback loop is the controlled feedback loop. If the robot and the environment are not in contact, then the dynamic behavior of the system reduces to the one represented by equation (1) (with \( d = 0 \), which is a plain positioning system. When the robot and the environment are in contact, then the values of the contact force and the end-point position of robot are given by \( f \) and \( y \) where the following equations are true:

\[
    y = G(e) + S(-f)
\]

(10)

\[
    e = r - H(f)
\]

(11)

If the operators in equations (10), (11), and (12) are considered as transfer function matrices, equations (13) and (14) can be obtained to represent the interaction force and the robot endpoint position for linearly treated systems when \( x_0 = 0 \).

\[
    f = E(I_n + S E + G H E)^{-1} G r
\]

(13)

\[
    y = (I_n + S E + G H E)^{-1} G r
\]

(14)

The objective is to choose a class of compensators, \( H \), to control the contact force with the input command \( r \). By knowing \( S \), \( G \), \( E \), and choosing \( H \), one can shape the contact force. The value of \( H \) is the choice of designer and, depending on the task, it can have various values in different directions. A large value for \( H \) develops a compliant robot while a small \( H \) generates a stiff robot. Reference [7] describes a micro manipulator in which the compliancy in the system is shaped for metal removal application. Note that \( S \) and \( GH \) add in equation (13) to develop the total compliancy in the system. \( GH \) represents the electronic compliancy in the robot while \( S \) models the natural hardware compliancy (such as RCC or the robot structural compliancy) in the system.\(^7\) Equation (13) also shows that a robot with good tracking capability (small

\(^7\)Equation (13) can be rewritten as \( f = (E^{-1} + S + GH)^{-1} G r \). Note that the environment admittance (1/impedance in the linear domain), \( E^{-1} \), the robot sensitivity (1/ stiffness in the linear domain), \( S \), and the electronic compliancy, \( GH \), add together to form the total sensitivity of the system. If \( H = 0 \), then only the admittance of the environment and the robot add together to form the compliancy for the system. By closing the loop via \( H \), one can not only add to the total sensitivity but also shape the sensitivity of the system.
gain for $S$ may generate a large contact force in a particular contact. One cannot choose arbitrarily large values for $H$; the stability of the closed-loop system of Fig. 3 must be guaranteed. The trade-off between the closed-loop stability and the size of $H$ is investigated in Section 6.

When the robot is not in contact with the environment (i.e., the outer feedback loops in Fig. 3 do not exist), the actual position of the robot end-point is governed by equation (1) (with $d = 0$). When the robot is in contact with the environment, then the contact force $r$ according to equations (10), (11), and (12). The input command vector, $r$, is used differently for the two categories of maneuverings; as an input trajectory command in unconstrained space (equation (1) with $d = 0$) and as a command to control force in constrained space. There is no hardware or software switch in the control system when the robot travels between unconstrained space and constrained space. The feedback loop on the contact force closes naturally when the robot encounters the environment.

6 Stability Analysis

The objective of this section is to arrive at a sufficient condition for stability of the system shown in Fig. 3. This sufficient condition leads to the introduction of a class of compensators, $H$, that can be used to develop compliancy for the family of robot manipulators with dynamic behavior represented by equation (1). Using operator $V$ defined by equation (9), the block diagram of Fig. 4 is constructed as a simplified version of the block diagram in Fig. 3. First we use the Small Gain Theorem to derive the general stability condition. Then, with the help of a corollary, we show the stability condition when $H$ is chosen as a linear operator (transfer function matrix) while $V$ is a nonlinear operator. Finally, if all the operators in Fig. 3 are transfer function matrices, then the stability bound is shown by inequality 25. Section 7 is devoted to stability analysis of the linearly treated systems, when the environment is infinitely rigid in comparison with the robot stiffness.

The following proposition (using the Small Gain Theorem in references [15, 16]) states the stability condition of the closed-loop system shown in Fig. 4.

If conditions I, II, and III hold:

I. $V$ is a $L^p_p$-stable operator, that is

(a) $V(e) : L^n_p \rightarrow L^n_p$  (15)

(b) existence of $\alpha \geq 0$ such that $\|V(e)\|_p \leq \alpha$  (16)

II. $H$ is chosen such that mapping $H(f)$ is $L^p_p$-stable, that is

(a) $H(f) : L^n_p \rightarrow L^n_p$  (17)

(b) existence of $\beta \geq 0$ such that $\|H(f)\|_p \leq \beta$  (18)

III. and $\alpha \alpha \beta < 1$  (19)

then the closed-loop system (Fig. 4) is $L^p_p$-stable. The proof is

The stability analysis and the role of robot sensitivity and environment dynamics on size $H$ are best shown by linear theory in equations (27)-(31). In particular, we confine our analysis to linear one-degree-of-freedom robot in equations (32) and (33) for better understanding of the nature of the stability analysis.

To guarantee the stability of the closed-loop system, $H$ must be chosen such that its “size” is smaller than the reciprocal of the “gain” of the forward loop mapping in Fig. 4. Note that $\gamma$ represents a “size” of $H$ in the singular value sense. When all the operators of Fig. 4 are linear transfer function matrices one can use Multivariable Nyquist Criterion (9) to arrive at the sufficient condition for stability of the closed loop system. This sufficient condition leads to the introduction of a class of transfer function matrices, $H$, that stabilize the family of linearly treated robot manipulators and environment using dynamic equations (2) and (4). The detailed derivation for the stability condition is given in Appendix C. Appendix D shows that the stability condition given by Nyquist Criterion is a subset of the condition given by the Small Gain Theorem. According to the results of Appendix C, the sufficient condition for stability is given by inequality 25.

$$\sigma_{\max}(GHE) < \sigma_{\min}(SE + I_p)$$  (20)

or a more conservative condition,

$$\sigma_{\max}(H) < \sigma_{\max}(E (I_p + S E)^{-1} G)$$  (24)

Fig. 4 Simplified version of Fig. 3. In the linear domain, $V = (E S + I_p)^{-1} G$. The trade-off between the closed-loop stability and the size of $H$ is investigated in Section 6.
stability criterion. The stability criterion when \( n = 1 \) is given by inequality 27.

\[
|HG| \leq \frac{1}{1+E} \quad \text{for all } \omega \in (0, \infty)
\]

(27)

where \( 1 \cdot 1 \) denotes the magnitude of a transfer function. Since in many cases \( G = 1 \) for all \( 0 \leq \omega \leq \omega_p \), then \( H \) must be chosen such that:

\[
|H| < \frac{1}{1+E} \quad \text{for all } \omega \in (0, \omega_p)
\]

(28)

Inequality 28 reveals some facts about the size of \( H \). The smaller the sensitivity of the robot manipulator is, the smaller \( H \) must be chosen. Also the inequality 28, the more rigid the environment is, the smaller \( H \) must be chosen. In the “ideal case,” no \( H \) can be found to allow a perfect positioning system \((S=0)\) to interact with an infinitely rigid environment \((E = \infty)\). In other words, for stability of the system shown in Fig. 3, there must be some compliancy either in robot or in the environment. RCC, structural dynamics and the tracking controller stiffness form the compliancy on the robot. Section 7 gives more information about the effects of \( E \) on the stability region.

7 Stability for Very Rigid Environment

In most manufacturing tasks, the end-point of the robot manipulator is in contact with a very stiff environment. Robotic deburring and grinding are examples of practical tasks in which the robot is in contact with stiff environment [6, 7]. According to the results in Appendix B, when the environment is very stiff, \((E=\infty)\), \((S=0)\), to interact with an infinitely rigid environment \((E = \infty)\). In other words, for stability of the system shown in Fig. 3, there must be some compliancy either in robot or in the environment. RCC, structural dynamics and the tracking controller stiffness form the compliancy on the robot. Section 7 gives more information about the effects of \( E \) on the stability region.

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8 An Example on the Stability Criteria

Consider a one-degree-of-freedom robot with \( G \) and \( S \) in equation (1) given as:

\[
G(s) = \frac{s + 1}{(s + 5)(s + 10 + 1)}
\]

\[
S(s) = \frac{s + 0.05}{s + 5 + 1}
\]

The system has a good positioning capability (small gain for \( S \) and unity gain for \( G \) at DC). The poles that are located at \(-250 \) and \(-300 \) show the high frequency modes in the robot. The stability of this system when it is in contact with various environment dynamics is analyzed. We assume \( E \) is constant and has the value of 10 for all frequency ranges. If we consider \( H \) as a constant gain, then inequality 27 yields that for \( H \leq 0.14 \) the value of \( |GH| \) is always smaller than \( |S + 1/E| \) for all \( \omega \in (0, \infty) \). Figure 5 shows the plots of \( |GH| \) and \( |S + 1/E| \) for three values of \( H \). For \( H = 0.08 \) the system is stable with the closed-loop poles located at \((-301.59, -244.81, -204.27, -9.25, -5.35, -7.37 \pm 8.4j) \). For \( H = 2.6 \) results in unstable system with the closed-loop poles located at \((-324.9, -221.31 \pm 63.5j, 0.78 \pm 37.82j, -9.01, -5.02) \). Note that the stability condition derived with inequality 27 is a sufficient condition for stability; many compensators can be found to stabilize the system without satisfying inequality 27. Figure 5 shows an example (\( H = 1.5 \)) that does not satisfy inequality 27 however the system is stable with closed-loop poles at \((-317.67, -221.66 \pm 49.06j, -2.48 \pm 29.9j, -9.02, -5.02) \). If one uses root locus for stability analysis, for \( H \geq 2.32 \) all the closed loop poles will be in the left half plane. Once a constant value for stabilizing \( H \) established, one can choose a dynamic compensator to filter out the high frequency noise in the force measurements:

\[
H = \frac{0.08}{s + 1}
\]
tain any spring or dampers. The compliancy in the active end-effector is developed electronically and therefore can be modulated by an on-line computer. Satisfying a kinematic constraint for this end-effector allows for uncoupled dynamic behavior for a bounded range. Two state-of-the-art miniature actuators power the end-effector directly. A miniature force cell measures the forces in two dimensions. The tool holder can maneuver a very light pneumatic grinder in a linear work-space of about 0.3 x 0.3 in. A bound for the global stability of the manipulator and environment has been derived. For stability of the environment and the robot taken as a whole, there must be some initial compliancy either in the robot or in the environment. The initial compliancy in the robot can be obtained by a nonzero sensitivity function for the positioning controller or a passive compliant element.

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References

APPENDIX A
Definitions 1 to 7 will be used in the stability proof of the closed-loop system (Vidyasagar, 1978, Vidyasagar and Desoer, 1975).

Definition 1: For all $p \in (1, \infty)$, we label as $L^p_n$ the set consisting of all functions $f = (f_1, f_2, \ldots, f_n)^T : (0, \infty) \to \mathbb{R}^n$ such that:

$$
\int_0^\infty |f_i|^p \, dt < \infty \quad \text{for } i = 1, 2, \ldots, n
$$

Definition 2: For all $T \in (0, \infty)$, the function $f_T$ defined by:

$$
f_T = \begin{cases} 
0 & 0 \leq t \leq T \\
t & T < t 
\end{cases}
$$

is called the truncation of $f$ to the interval $(0, T)$.

Definition 3: The set of all functions $f = (f_1, f_2, \ldots, f_n)^T : (0, \infty) \to \mathbb{R}^n$ such that $f \in L^p_n$ for all finite $T$ is denoted by $L^p_{\infty}$.

Definition 4: The norm on $L^p_n$ is defined by:

$$
\|f\|_p = \left( \int_0^\infty |f|^p \, dt \right)^{1/p}
$$

Definition 5: Let $v(\cdot) : L^p_{\infty} \to L^p_{\infty}$. We say that the operator $V(\cdot)$ is $L^p$-stable, if:

(a) $v(\cdot) : L^p_n \to L^p_n$

(b) there exist finite real constants $\alpha_4$ and $\beta_4$ such that:

$$
\|V(e)\|_p \leq \alpha_4 \|e\|_p + \beta_4 \quad \forall e \in L^p_n
$$

According to this definition we first assume that the operator maps $L^p_{\infty}$ to $L^p_{\infty}$. It is clear that if one does not show that $v(\cdot) : L^p_{\infty} \to L^p_{\infty}$, the satisfaction of condition (a) is impossible since $L^p_{\infty}$ contains $L^p_n$. Once mapping, $v(\cdot)$, from $L^p_{\infty}$ to $L^p_{\infty}$ is established, then we say that the operator $v(\cdot)$ is $L^p_n$-stable if, whenever the input belongs to $L^p_{\infty}$, the resulting output belongs to $L^p_n$. Moreover, the norm of the output is not larger than $\alpha_4$ times the norm of the input plus the offset constant $\beta_4$.

Definition 6: The smallest $\alpha_4$ such that there exist a $\beta_4$ so that inequality b of Definition 5 is satisfied is called the gain of the operator $v(\cdot)$.

Definition 7: Let $V(\cdot) : L^p_{\infty} \to L^p_{\infty}$. The operator $V(\cdot)$ is said to be causal if:

$$
V(e) = V(e_T) \quad T < \infty \quad \text{and} \quad \forall e \in L^p_{\infty}
$$

Proof of the Nonlinear Stability Proposition. Define the closed-loop mapping $A : r \to e$ (Fig. 4).

$$
e = r - H(V(e))
$$

(A1)

For each finite $T$, inequality (A2) is true.

$$
\|e_T\|_p \leq \|e_T\|_p + \|H(V(e))\|_p \quad \forall e \in (0, T) (A2)
$$

Since $H(V(e))$ is $L^p$-stable. Therefore, inequality (A3) is true.

$$
\|e_T\|_p \leq \|r_T\|_p + \alpha_4 \|e_T\|_p + \alpha_5 \beta_4 + \beta_5
$$

for all $e \in (0, T)$

(A3)

Since $\alpha_4 \alpha_5$ is less than unity:

$$
\|e_T\|_p \leq \|r_T\|_p \frac{1}{1 - \alpha_4 \alpha_5} + \frac{\alpha_5 \beta_4 + \beta_5}{1 - \alpha_4 \alpha_5} \quad \forall e \in (0, T)
$$

(A4)

Inequality (A4), shows that $e(\cdot)$ is bounded over $(0, T)$. Because this reasoning is valid for every finite $T$, it follows that $e(\cdot) \in L^p_{\infty}$, i.e., that $A : L^p_{\infty} \to L^p_{\infty}$. Next we show that the mapping $A$ is $L^p_{\infty}$-stable in the sense of Definition 5. Since $r \in L^p_{\infty}$, therefore $\|r\|_p < \infty$ for all $e \in (0, \infty)$, therefore in-
inequality (A5) is true.
\[ |e| < \infty \quad \text{for all } t \in (0, \infty) \quad (A5) \]

inequality (A5) implies \( e \) belongs to \( L_2 \)-space whenever \( r \) belongs to \( L_p \)-space. With the same reasoning from equations (A1) to (A5), it can be shown that inequality (A6) is true.
\[ |e| \leq \frac{1}{1 - \alpha_3 \alpha_4} + \frac{\alpha_5 \beta_4}{1 - \alpha_3 \alpha_4} \quad \text{for all } t \in (0, \infty) \quad (A6) \]

Inequality (A6) shows the linear boundedness of \( e \) (condition b of definition S). Inequalities (A5) and (A6) taken together, guarantee that the closed-loop mapping \( A \) is \( L_p \)-stable.

**APPENDIX B**

A very rigid environment generates a very large force for a small displacement. We choose the minimum singular value of \( E \) to represent the size of \( E \). The following proposition states the limiting value of the force when the robot manipulator is in contact with a very rigid environment.

If \( u_{\text{min}}(E) > M_0 \), where \( M_0 \) is an arbitrarily large number, then the value of the force given by equation (13) will approach to the expression given by equation (B1).
\[ f_\omega = (S + GH)^{-1}(S + SE + GHE)^{-1}G \quad (B1) \]

Proof: We will prove that \( |f_\omega - f| \) approaches a small number as \( M_0 \) approaches a large number.
\[ f_\omega - f = (S + GH)^{-1}(l_\omega - (S + GHE)^{-1}G \quad (B2) \]

Factoring \( (l_\omega + SE + GHE)^{-1} \) to the right-hand side:
\[ f_\omega - f = (S + GH)^{-1}(l_\omega + SE + GHE)^{-1}G (B3) \]

If \( |f_\omega - f| < |\sigma_{\max}(S + GH)|^{-1} \times |\sigma_{\max}(l_\omega + SE + GHE)^{-1}G| \quad (B4) \]

If \( |f_\omega - f| < \frac{1}{\sigma_{\min}(S + GH)} \times \sigma_{\max}(S + SE + GHE)^{-1} \quad (B5) \]

If \( |f_\omega - f| < \frac{1}{\sigma_{\min}(S + GH)} \times \sigma_{\max}(S + SE + GHE)^{-1} \quad (B6) \]

\( \sigma_{\max}(G) \) and \( \sigma_{\min}(S + GH) \) are bounded values. If \( \sigma_{\min}(E) > M_0 \), then it is clear that the left-hand side of inequality (B6) can be an arbitrarily small number by choosing \( M_0 \) to be a large number. The proof for \( y_\omega = 0 \) is similar to the above.

**APPENDIX C**

The objective is to find a sufficient condition for stability of the closed-loop system in Fig. 3 by Nyquist Criterion. The block diagram in Fig. 3 can be reduced to the block diagram in Fig. C1 when all the operators are linear transfer function matrices and \( \gamma_0 = 0 \).

There are two elements in the feedback loop; \( GHE \) and \( SE \). \( SE \) shows the natural force feedback while \( GHE \) represents the controlled force feedback in the system. If \( H = 0 \), then the system in Fig. C1 reduces to the system in Fig. 2 (a stable positioning robot manipulator which is in contact with the environment \( E \)). The objective is to use Nyquist Criterion (9) to arrive at the sufficient condition for stability of the system when \( H \neq 0 \). The following conditions are regarded:

1) The closed loop system in Fig. C1 is stable if \( H = 0 \). This condition simply states the stability of the robot manipulator and environment when they are in contact. (Fig. 2 shows this configuration.)

2) \( H \) is chosen as a stable linear transfer function matrix.

Therefore the augmented loop transfer function \( (GHE + SE) \) has the same number of unstable poles that \( SE \) has. Note that in many cases \( SE \) is a stable system.

3) Number of poles on \( j\omega \) axis for both loop \( SE \) and \( (GHE + SE) \) are equal.

Considering that the system in Fig. C1 is stable when \( H = 0 \), we plan to find how robust the system is when \( GHE \) is added to the feedback loop. If the loop transfer function \( SE \) (without compensator, \( H \)) develops a stable closed-loop system, then we are looking for a condition on \( H \) such that the augmented loop transfer function \( (GHE + SE) \) guarantees the stability of the closed-loop system. According to the Nyquist Criterion, the system in Fig. C1 remains stable if the anti-clockwise encirclement of the det \( (SE + GHE + I_n) \) around the center of the \( s \)-plane is equal to the number of unstable poles of the loop transfer function \( (GHE + SE) \). According to conditions 2 and 3, the loop transfer functions \( SE \) and \( (GHE + SE) \) both have the same number of unstable poles. The closed-loop system when \( H = 0 \) is stable according to condition 1; the encirclements of det \( (SE + I_n) \) is equal to unstable poles of \( SE \). When \( GHE \) is added to the system, for stability of the closed-loop system, the number of the encirclements of det \( (SE + GHE + I_n) \) must be equal to the number of unstable poles of the \( (GHE + SE) \). Since the number of unstable poles of \( (SE + GHE) \) and \( SE \) are the same, the stability of the system det \( (SE + GHE + I_n) \) must have the same number of encirclements that det \( (SE + I_n) \) has. A sufficient condition to guarantee the equality of the number of encirclements of det \( (SE + GHE + I_n) \) and det \( (SE + I_n) \) is that the det \( (SE + GHE + I_n) \) does not pass through the origin of the \( s \)-plane for all possible nonzero but finite values of \( H \), or
\[ \text{det}(SE + GHE + I_n) \neq 0 \quad \text{for all } \omega \in (0, \infty) \quad (C1) \]

If inequality C1 does not hold then there must be a nonzero vector \( z \) such that:
\[ (SE + GHE + I_n)z = 0 \quad (C2) \]

or:
\[ GHE z = -(SE + I_n)z \quad (C3) \]

A sufficient condition to guarantee that equality (C3) will not happen is given by inequality (C4).
\[ \sigma_{\max}(GHE) < \sigma_{\min}(SE + I_n) \quad \text{for all } \omega \in (0, \infty) \quad (C4) \]

or a more conservative condition:
\[ \frac{1}{\sigma_{\max}(E(SE + I_n)^{-1}G)} < \frac{1}{\sigma_{\min}(SE + I_n)} \quad \text{for all } \omega \in (0, \infty) \quad (C5) \]

Note that \( E(SE + I_n)^{-1}G \) is the transfer function matrix that maps \( e \) to the contact force, \( f \). Figure 4 shows the closed-loop system. According to the result of the proposition, \( H \) must be chosen such that the size of \( H \) is smaller than the reciprocal of the size of the forward loop transfer function, \( E(SE + I_n)^{-1}G \).

**APPENDIX D**

The following inequalities are true when \( p = 2 \) and \( H \) and \( V \) are linear operators.
\[ ||H(V(e))||_p \leq \sigma_{\max}(SE + I_n) ||V(e)||_p \quad (D1) \]
\[ ||V(e)||_p \leq \mu \cdot ||e||_p \quad (D2) \]
where:
\[ p = \sigma_{\text{max}}(Q), \text{ and } Q \text{ is the matrix whose } i/j \text{th entry is given by } (Q)_{ij} = \sup_{\omega \in \omega} \| V \|_{ij}, \]
\[ \nu = \sigma_{\text{max}}(R), \text{ and } R \text{ is the matrix whose } i/j \text{th entry is given by } (R)_{ij} = \sup_{\omega \in \omega} \| H \|_{ij}. \]

Substituting inequality (D2) in (D1):
\[ \| IHV(\omega) \|_{p} \leq \mu \nu \| \omega \|_{p} \tag{D3} \]

According to the stability condition, to guarantee the closed loop stability, \( \mu \nu < 1 \):
\[ \nu < \frac{1}{\mu} \tag{D4} \]

Note that the followings are true:
\[ \sigma_{\text{max}}(V) \leq \mu \text{ for all } \omega \in (0, \infty) \tag{D5} \]
\[ \sigma_{\text{max}}(H) \leq \nu \text{ for all } \omega \in (0, \infty) \tag{D6} \]

Substituting (D5) and (D6) into inequality (D4) which guarantees the stability of the system, the following inequality is obtained:
\[ \sigma_{\text{max}}(H) < \frac{\nu}{\sigma_{\text{max}}(V)} \text{ for all } \omega \in (0, \infty) \tag{D7} \]
\[ \sigma_{\text{max}}(H) < \frac{\nu}{\sigma_{\text{max}}(E(l_4 + S_4)^{-1} G)} \text{ for all } \omega \in (0, \infty) \tag{D8} \]

Inequality (D8) is identical to inequality (26). This shows that the linear condition for stability given by the multivariable Nyquist Criterion is a subset of the general condition given by the Small Gain Theorem.

**APPENDIX E**

This Appendix is dedicated to deriving of the Jacobian and the mass matrix of a general five-bar linkage. In Fig. E1, \( J_1, l_1, x_1, m_1, \) and \( \theta_1 \) represent the moment of inertia relative to the end-point, length, location of the center of mass, mass and the orientation of each link for \( i = 1, 2, 3 \) and 4.

Using the standard method, the Jacobian of the linkage can be represented by equation (E1).

\[ J_c = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{E1} \]

where:
\[ J_{11} = -l_1 \sin(\theta_1) + a_1 l_2 \sin(\theta_2), \quad J_{12} = l_1 \cos(\theta_1) - a_1 l_2 \cos(\theta_2) \]
\[ J_{12} = b_1 l_1 \sin(\theta_2), \quad J_{22} = b_1 l_2 \cos(\theta_2) \]

The mass matrix is given by equation (E2).

\[ M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{E2} \]

where:
\[ M_{11} = J_1 + m_2 l_1^2 + J_2 a_1^2 + J_3 c_3^2 + 2 x_2 l_1 \cos(\theta_1 - \theta_2) a_1 m_2 \]
\[ M_{12} = J_2 a_1 b_2 \cos(\theta_1 - \theta_2) x_2 l_1 l_2 + J_3 c_3 d_3 + m_3 l_3 \]
\[ M_{21} = M_{12} \]
\[ M_{22} = 2 m_1 l_4 x_3 d_3 \cos(\theta_4 - \theta_3) + J_1 d_1^2 + J_4 + m_3 l_4^2 + J_2 b_2^2 \]

\( a, b, c, d \) are given below.
\[ a = l_1 \sin(\theta_1 - \theta_2) / (l_2 \sin(\theta_2 - \theta_3)) \]
\[ b = l_4 \sin(\theta_4 - \theta_3) / (l_2 \sin(\theta_2 - \theta_3)) \]
\[ c = l_1 \sin(\theta_1 - \theta_2) / (l_3 \sin(\theta_2 - \theta_3)) \]
\[ d = l_4 \sin(\theta_4 - \theta_3) / (l_3 \sin(\theta_2 - \theta_3)) \]
the input command represents the magnitude of $G$ at each frequency. For measurement of the sensitivity transfer function matrix, the input excitation was supplied by the rotation of an eccentric mass mounted on the tool bit. The rotating mass exerts a centrifugal, sinusoidal force on the tool bit. The frequency of the imposed force is equal to the frequency of rotation of the mass. By varying the frequency of the rotation of the mass, one can vary the frequency of the imposed force on the end-effector. Figure 10 depicts the sensitivity transfer function. The values of the sensitivity transfer functions along the normal and tangential directions, within their bandwidths, are 0.7 in/lbf and 0.197 in/lbf, respectively.

The nature of compliancy for the end effector is given by equation (31). $H$ was chosen such that $(S + H)^{-1}$ in each direction is equal to the desired stiffness. $H$ must also guarantee the stability of the closed-loop system. The stability criteria for a one-degree-of-freedom system is given by inequalities 32 and 33. Inequality 33 shows that the more rigid the environment is, the smaller $H$ must be chosen to guarantee the stability of the closed-loop system. In the case of a rigid environment ("large" $E$) and a "good" positioning system ("small" $S$), $H$ must be chosen as a very small gain. The values for $H$ along the normal and tangential directions within their bandwidths are 0.01 in/lbf and 0.194 in/lbf, respectively. These values result in 0.39 in/lbf and 0.7 in/lbf for $(S + H)^{-1}$ within the bandwidth of the system. Figure 11 shows the experimental and theoretical values of the end-point compliancy (Fig. 11 actually shows the end-point admittance where it is reciprocal of the impedance in the linear case.)

In another set of experiments, the whole end-effector was moved in two different directions to encounter a edge of a part. The objective was to observe the uncoupled time-domain dynamic behavior of the end-effector when the end-effector is in contact with the hard environment. The controller was designed such that the values of $(S + H)^{-1}$ in tangential and normal direction are 0.32 lbf/in and 4.0 lbf/in, respectively. First the end-effector was moved 0.5 in. beyond the edge of the part in $y_n$-direction. Figure 12 shows the contact forces. The force in $y_n$-direction increases from zero to 2.0 lbf while the force in the $y_r$-direction remains at zero. Next the end-effector was moved 0.5 in. beyond the edge of the part in the $y_r$-direction. Figure 13 shows the contact forces. The force in $y_r$-direction increases from zero to 0.16 lbf while the force in $y_n$-direction remains at zero. In both cases the end-effector was moved as 0.5 in. beyond the edge of the stiff wall. Since the stiffness of the end-effector in $y_r$-direction is larger than the stiffness in $y_n$-direction, the contact force in $y_r$-direction is larger than the contact force in $y_n$-direction.

11 Summary and Conclusion

A new controller architecture for compliance control has been investigated using unstructured models for dynamic behavior of robot manipulators and environment. This unified approach of modeling robot and environment dynamics is expressed in terms of sensitivity functions. The control approach allows not only for tracking the input-command vector, but also for compliance in the constrained maneuverings. An active end-effector has been designed, built, and tested for verification of the control method. The active end-effector (unlike the passive system) does not con-
tain any spring or dampers. The compliance in the active end-effector is developed electronically and therefore can be modulated by an on-line computer. Satisfying a kinematic constraint for this end-effector allows for uncoupled dynamic behavior for a bounded range. Two state-of-the-art miniature actuators power the end-effector directly. A miniature force cell measures the forces in two dimensions. The tool holder can maneuver a very light pneumatic grinder in a linear work-space of about 0.3 x 0.3 in. A bound for the global stability of the manipulator and environment has been derived. For stability of the environment and the robot taken as a whole, there must be some initial compliance either in the robot or in the environment. The initial compliance in the robot can be obtained by a nonzero sensitivity function for the positioning controller or a passive compliant element.

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References


APPENDIX A

Definitions 1 to 7 will be used in the stability proof of the closed-loop system (Vidyasagar, 1978, Vidyasagar and Desoer, 1975).

Definition 1: For all p(1,∞), we label as $L^n_p$ the set consisting of all functions $f = (f_1, f_2, \ldots, f_n)^T: (0,\infty) \rightarrow \mathbb{R}^n$ such that:

$$\int_0^\infty |f_i|^p dt < \infty \quad \text{for } i = 1, 2, \ldots, n$$

Definition 2: For all $T \in (0, \infty)$, the function $f_T$ defined by:

$$f_T = \begin{cases} f & 0 \leq t \leq T \\ 0 & T < t \end{cases}$$

is called the truncation of $f$ to the interval $(0,T)$.

Definition 3: The set of all functions $f = (f_1, f_2, \ldots, f_n)^T: (0,\infty) \rightarrow \mathbb{R}^n$ such that $f \in L^n_p$ for all finite $T$ is denoted by $L^n_p$. For each finite $T$, Inequality (A2) is true.

Definition 4: Let $V(.): L^n_p \rightarrow L^n_p$. The operator $V(.)$ is called the truncation of $f$ to the interval $(0,T)$.

Definition 5: Let $V(.): L^n_p \rightarrow L^n_p$. We say that the operator $V(.)$ is $L^n_p$-stable if:

(a) $V(.) \in L^n_p$, and $V e \in L^n_p$

(b) there exist finite real constants $\alpha_4$ and $\beta_4$ such that:

$$|VV(e)|_p \leq \alpha_4 |e|_p + \beta_4 \quad \forall e \in L^n_p$$

According to this definition we first assume that the operator maps $L^n_p$ to $L^n_p$. It is clear that if one does not show that $V(.)L^n_p \subseteq L^n_p$, the satisfaction of condition (a) is impossible since $L^n_p$ contains $L^n_p$. Once mapping, $V(.)$, from $L^n_p$ to $L^n_p$ is established, then we say that the operator $V(.)$ is $L^n_p$-stable if, whenever the input belongs to $L^n_p$, the resulting output belongs to $L^n_p$. Moreover, the norm of the output is not larger than $\alpha_4$ times the norm of the input plus the offset constant $\beta_4$.

Definition 6: The smallest $\alpha_4$ such that there exist a $\beta_4$ so that inequality b of Definition 5 is satisfied is called the gain of the operator $V(.)$.

Definition 7: Let $V(.): L^n_p \rightarrow L^n_p$. The operator $V(.)$ is said to be causal if:

$$V(e) \tau = V(e_T) \quad \forall T < \infty \quad \text{and} \quad \forall e \in L^n_p$$

Proof of the Nonlinear Stability Proposition. Define the closed-loop mapping $A : r = e$ (Fig. 4).

$$e = r - H(V(e))$$

For each finite $T$, inequality (A2) is true.

$$|le \tau|_p \leq |le|_p + |lH(V(e))\tau|_p$$

Since $H(V(e))$ is $L^n_p$-stable. Therefore, inequality (A3) is true.

$$|le \tau|_p \leq |le|_p + \alpha_4 |e|_p + \alpha_5 |e|_p + \alpha_6 |e|_p$$

for all $\tau(0,T)$

Therefore, $\alpha_5|e|_p$ is less than unity:

$$|le \tau|_p \leq \frac{|le \tau|_p}{1 - \alpha_5} + \frac{\alpha_6}{1 - \alpha_4}$$

Inequality (A4), shows that $e(\cdot)$ is bounded over $(0,T)$. Because this reasoning is valid for every finite $T$, it follows that $e(\cdot) \in L^n_p$, i.e., $A: L^n_p \rightarrow L^n_p$. Next we show that the mapping $A$ is $L^n_p$-stable in the sense of definition 5. Since $re \in L^n_p$, therefore $|le|_p \leq \infty$ for all $\tau(0,\infty)$, therefore in-
equality (A5) is true.

\[ |l|_p < \infty \quad \text{for all } t \in (0, \infty) \quad \text{(A5)} \]

Inequality (A5) implies \( e \) belongs to \( L^p \)-space whenever \( r \) belongs to \( L^p \)-space. With the same reasoning from equations (A1) to (A5), it can be shown that inequality (A6) is true.

\[ |l|_p \leq \frac{|l|_p}{1 - \alpha \beta_0 + \beta_1} \quad \text{for all } t \in (0, \infty) \quad \text{(A6)} \]

Inequality (A6) shows the linear boundedness of \( e \) (condition b of definition 5). Inequalities (A5) and (A6) taken together, guarantee that the closed-loop mapping \( A \) is \( L^p \)-stable.

**APPENDIX B**

A very rigid environment generates a very large force for a small displacement. We choose the minimum singular value of \( E \) to represent the size of \( E \). The following proposition states the limiting value of the force when the robot manipulator is in contact with a very rigid environment.

If \( \sigma_{\min}(E) > M_0 \), where \( M_0 \) is an arbitrarily large number, then the value of the force given by equation (13) will approach to the expression given by equation (B1)

\[ f_\infty = (S + GH)^{-1} f \quad \text{(B1)} \]

Proof: We will prove that \( |f_\infty - f| \) approaches a small number as \( M_0 \) approaches a large number.

\[ |f_\infty - f| < |\sigma_{\max}(S + GH)^{-1} (I + SE + GHE)^{-1} G r| \quad \text{(B2)} \]

Factoring \((I_\infty + SE + GHE)^{-1}\) to the right-hand side:

\[ |f_\infty - f| < |\sigma_{\max}(S + GH)^{-1} (I_\infty + SE + GHE)^{-1} G r| \quad \text{(B3)} \]

\[ |f_\infty - f| < \frac{\sigma_{\max}(G)}{\sigma_{\min}(S + GH) \times \sigma_{\min}(SE + GHE) - 1} \quad \text{(B4)} \]

\[ \sigma_{\max}(G) \text{ and } \sigma_{\min}(S + GH) \text{ are bounded values. If } \sigma_{\min}(E) > M_0, \text{ then it is clear that the left-hand side of inequality (B6) can be an arbitrarily small number by choosing } M_0 \text{ to be a large number. The proof for } y_\infty = 0 \text{ is similar to the above.} \]

\[ \text{(B6)} \]

**APPENDIX C**

The objective is to find a sufficient condition for stability of the closed-loop system in Fig. 3 by Nyquist Criterion. The block diagram in Fig. 3 can be reduced to the block diagram in Fig. C1 when all the operators are linear transfer function matrices and \( \sigma_0 = 0 \).

There are two elements in the feedback loop: \( GHE \) and \( SE \). \( SE \) shows the natural force feedback while \( GHE \) represents the controlled force feedback in the system. If \( H = 0 \), then the system in Fig. C1 reduces to the system in Fig. 2 (a stable positioning robot manipulator which is in contact with the environment \( E \)). The objective is to use Nyquist Criterion (9) to arrive at the sufficient condition for stability of the system when \( H \neq 0 \). The following conditions are regarded:

1) The closed loop system in Fig. C1 is stable if \( H = 0 \). This condition simply states the stability of the robot manipulator and environment when they are in contact. (Fig. 2 shows this configuration.)

2) \( H \) is chosen as a stable linear transfer function matrix.

\[ \text{(B6)} \]

\[ \sigma_{\max}(GHE) < \sigma_{\min}(SE + I_\infty) \quad \text{for all } \omega \in (0, \infty) \quad \text{(C3)} \]

A sufficient condition to guarantee that equality (C3) will not happen is given by inequality (C4).

\[ \sigma_{\max}(GHE) < \sigma_{\min}(SE + I_\infty) \quad \text{for all } \omega \in (0, \infty) \quad \text{(C4)} \]

or a more conservative condition:

\[ \sigma_{\max}(H) < \frac{1}{\sigma_{\max}(E(SE + I_\infty)^{-1} G)} \quad \text{for all } \omega \in (0, \infty) \quad \text{(C5)} \]

\[ \text{Note that } E(SE + I_\infty)^{-1} G \text{ is the transfer function matrix that maps } e \text{ to the contact force, } f. \text{ Figure 4 shows the closed-loop system. According to the result of the proposition, } H \text{ must be chosen such that the size of } H \text{ is smaller than the reciprocal of the size of the forward loop transfer function, } E(SE + I_\infty)^{-1} G. \]

**APPENDIX D**

The following inequalities are true when \( p = 2 \) and \( H \) and \( V \) are linear operators.

\[ ||H(V(e))||_p \leq \mu ||V(e)||_1 \quad \text{(D1)} \]

\[ ||V(e)||_1 \leq \mu ||e||_p \quad \text{(D2)} \]
where:

\[ \mu = \sigma_{\text{max}}(Q) \]

and \( Q \) is the matrix whose \( ij \)th entry is given by \( (Q)_{ij} = \sup_{\omega} \|V\|_p \).

\[ \nu = \sigma_{\text{max}}(R) \]

and \( R \) is the matrix whose \( ij \)th entry is given by \( (R)_{ij} = \sup_{\omega} \|H\|_p \).

Substituting inequality (D2) in (D1):

\[ \|HV(e)\|_p \leq \mu \|e\|_p \]  \( \text{(D3)} \)

According to the stability condition, to guarantee the closed loop stability \( \mu \nu < 1 \) or:

\[ \nu < \frac{1}{\mu} \]  \( \text{(D4)} \)

Note that the followings are true:

\[ \sigma_{\text{max}}(V) \leq \mu \]

for all \( \omega \in (0, \infty) \)  \( \text{(D5)} \)

\[ \sigma_{\text{max}}(H) \leq \nu \]

for all \( \omega \in (0, \infty) \)  \( \text{(D6)} \)

Substituting (D5) and (D6) into inequality (D4) which guarantees the stability of the system, the following inequality is obtained:

\[ \sigma_{\text{max}}(H) < \frac{1}{\sigma_{\text{max}}(V)} \]

for all \( \omega \in (0, \infty) \)  \( \text{(D7)} \)

\[ \sigma_{\text{max}}(H) < \frac{1}{\sigma_{\text{max}}(E(I_x + SE)^{-1}G)} \]

for all \( \omega \in (0, \infty) \)  \( \text{(D8)} \)

Inequality (D8) is identical to inequality (26). This shows that the linear condition for stability given by the multivariable Nyquist Criterion is a subset of the general condition given by the Small Gain Theorem.

**APPENDIX E**

This Appendix is dedicated to deriving of the Jacobian and the mass matrix of a general five-bar linkage. In Fig. E1, \( l_i, l, x_i, \theta, \) and \( \theta_i \) represent the moment of the inertia relative to the end-point, length, location of the center of mass, mass and the orientation of each link for \( i = 1, 2, 3 \) and 4.

Using the standard method, the Jacobian of the linkage can be represented by equation (E1).

\[ J_e = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \]

where:

\[ J_{11} = -l_1 \sin(\theta_1) + a l_3 \sin(\theta_2), \quad J_{21} = l_1 \cos(\theta_1) - a l_3 \cos(\theta_2) \]

\[ J_{12} = b l_3 \sin(\theta_2), \quad J_{22} = b l_3 \cos(\theta_2) \]

The mass matrix is given by equation (E2).

\[ M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \]

where:

\[ M_{11} = l_1 + m_2 l_1^2 + J_2 a^2 + J_3 c^2 + 2 x_2 l_1 \cos(\theta_1 - \theta_2) a m_2 \]

\[ M_{12} = J_2 a b + b \cos(\theta_1 - \theta_2) x_2 l_1 m_3 + J_3 c d \]

\[ + c \cos(\theta_4 - \theta_3) x_3 l_4 m_3 \]

\[ M_{21} = M_{12} \]

\[ M_{22} = 2 m_3 l_4 x_3 d \cos(\theta_4 - \theta_3) + J_3 a d^2 + J_4 + m^2 l_3^2 + J_5 b^2 \]

\[ a, b, c, d \] are given below.

\[ a = l_1 \sin(\theta_1 - \theta_2)/(l_2 \sin(\theta_2 - \theta_3)) \]

\[ b = l_4 \sin(\theta_4 - \theta_3)/(l_2 \sin(\theta_2 - \theta_3)) \]

\[ c = l_1 \sin(\theta_1 - \theta_2)/(l_1 \sin(\theta_2 - \theta_3)) \]

\[ d = l_4 \sin(\theta_4 - \theta_3)/(l_3 \sin(\theta_2 - \theta_3)) \]
All parameters were initially assumed to be set to their true value, but the value of mass at the end of the second link, $m_2$ was increased from 2 kg to 3 kg at the time $t = 5$ sec. Figures 3 to 6 show the results of the tracking error response and the parameter identification process. $\Gamma_{33} = 1$ was used for all the cases shown. Note that the state variables (Figs. 3, 4) have essentially come to their correct values within three seconds. As expected, the estimates of the second mass (Fig. 6) show the most severe estimation errors; however this estimate has also converged to the actual mass value after three seconds. The estimate of the first mass (Fig. 5) shows a maximum error of approximately 10 percent from the actual value.

5 Conclusions

From the above developments, we can draw the following conclusions: First, the adaptive pure computed torque algorithm is easy to implement, computationally very efficient and is stable under moderate feedback gains. The algorithm allows accurate estimations of system parameters and excellent control of the system state variables. The design is made very simple by the explicit expression of the parameter range allowed for stability. Second, if no additional filters are used, and a Lyapunov function method is assumed, then this paper seems to have accounted for many of the reasonable choices of the control laws of the form (5) and adaptation laws in the form of (9).

References